Economics Education and Research Consortium
Working Paper Series

Analysis of Short-Term Equilibria in a Housing Market with Application to Development of Housing Policy

Alexander Khutoretsky

Working Paper No 2K/05

This project (No 98-278) was supported
by the Economics Education and Research Consortium

Research area: **Enterprises and Product Markets**

JEL Classification: D51, H42, H53, R21, R38

KHUTORETSKY A.B. Analysis of Short-Term Equilibria in a Housing Market with Application to Development of Housing Policy — Moscow: EERC, 2001 — pp 1 – 67.

This paper concentrates on the theoretical analysis of local housing market behavior in a short-term period such that the set of market agents, the set of dwellings, and the agents' preferences are invariable within this period. The basic model presents the market as an economy of exchange with quasi-linear utility functions of market agents. The model covers the sectors of dwellings for sale and for rent and takes into account the indivisibility of dwellings. Walrasian equilibria (their complete description, efficiency, and comparative statics) and equilibria with respect to rationing schemes (their existence, uniqueness, and efficiency) are studied. A hypothetical market mechanism ensuring transition to an equilibrium from an arbitrary initial situation is formulated. Based on the obtained results, some approaches to regulation of housing markets are pointed out and the corresponding methods described.

Acknowledgements. The author is indebted to EERC experts M. Alexeev, M. Schaffer, E.V. Serova, and especially to R.M. Entov and R. Ericson for fruitful recommendations and help in refining the economic basis of the market model. The author is obliged also to R. Arnott, V.P. Busygin, V.P. Fjodorov, K.P. Glustchenko, G. Gutin, E.B. Kibalov, N.B. Kosareva, O.S. Pchelintsev, and P. Thalmann for useful discussions and advice. The author gratefully acknowledges the support of the Open Society Institute during the first stages of work.

Keywords: Russia, housing market, indivisibility, equilibrium, rationing scheme, market mechanism, regulation, linear programming.

Alexander Khutoretsky
Institute of Economics and Industrial Engineering, Siberian Branch of RAS
17, Acad. Lavrentiev pr., 630090 Novosibirsk, Russia
Phone: +7 (383 2) 34 1350, (383 2) 30 2548
Fax: +7 (383 2) 30 2580
E-mail: hab@dus.nsc.ru

CONTENTS

NON-TECHNICAL SUMMARY

A housing market allows each household to select a dwelling that most closely suits its needs, within the framework of budget limitations. If a household cannot allocate enough means to purchase housing, then its housing conditions fall below the socially acceptable level. The state can adjust the housing market in order to minimize the number of households with socially unacceptable housing and, as a rule, such regulation is more or less being implemented.

In this paper, we analyze only one aspect of the housing market regulation problem. Namely, we consider two alternatives for regulating a local housing market in the short-run: state housing construction (regulation from the supply side) and housing subsidies (regulation from the demand side). Our purpose is to suggest and to substantiate some methods for achieving optimal (or quasi-optimal) regulatory action in housing markets.

Detailed literature comments are incorporated into the body of the paper; here we only explain the logic employed in the study and the main results.

It is natural to suppose that the expediency of a regulatory action (a program directed towards either municipal housing construction or subsidizing market agents) can be evaluated using the allocation of dwellings and the price structure in the equilibrium attained after and, partly, owing to realization of this action. This assumption is true if the following conditions are met: (a) equilibria exist and they are effective; (b) some mechanism intrinsic to housing markets leads them to equilibrium from any initial situation. Therefore, theoretical analysis of the equilibria in a housing market is indispensable and a significant part of the paper is devoted to this topic.

First of all, in Section 1 we formulate a plausible model of a housing market for the short-term. We assume that neither new households nor new dwellings emerge and that market agents' preferences do not change during this period. The model presents the market as a monetary exchange economy and takes into account the indivisibility of dwellings. The model covers two sectors of the market: dwellings for sale and dwellings for rent, with an invariable distribution of dwellings between them.

It is known that in a market model of this type, an equilibrium exists and that there can be many equilibria. Therefore, we need a convenient description of all possible equilibria for the given initial situation. Such a description (in terms of solutions and dual prices of some linear programming problem) is presented in Section 2.1. It is then proved that the core of the market coincides with the set of all equilibrium distributions and is a proper subset of the set of all Pareto-maximal distributions. Hence, competitive equilibria are the desirable (efficient) states of the given market. Also, some properties of the equilibria, for example, that each consumer has an acceptable level of non-housing consumption, are proved in this section.

The numerousness of equilibria in the considered model is the rule rather than the exception. Thus, it would not be enough only to show that the market necessarily will come to some equilibrium from any initial situation; we also have to understand how the market "selects" one equilibrium out of the myriad possibilities. For this reason, we construct some rationing schemes in Section 2.2 (we believe that the analogous schemes are inherent to housing markets in the short-run). These schemes create the demand and/or supply constraints perceived by market agents under a system of fixed prices. Consumers enter the market in some order and each "examines" the dwellings available to him or her in some order. It is these orderings that determine the specific rationing scheme and each scheme creates a unique and efficient fixed-price equilibrium.

In Section 3, we then construct a hypothetical, but plausible, market mechanism for ensuring transition to equilibrium through a finite sequence of fixed-price equilibria with respect to a rationing scheme. The specific rationing scheme determines the resulting competitive equilibrium.

The above mechanism of market equilibrating explains some features of housing market activity. For example, it is not unusual in many local markets for there to be many dwellings for sale that are not in demand given current prices, yet the prices do not come down. Does this mean that the market is not in equilibrium? The answer is negative. First, if the current price of a dwelling owned by a supplier is lower than his/her reservation price, then the supplier receives maximum utility by taking this dwelling off the market. Second, a reduction in the price of a dwelling owned by a consumer does not change the utility of this dwelling for him/her, but only moderates the utility he/she can receive when choosing some other dwelling. Therefore, at some base price, the best choice for an agent is to keep the dwelling and he/she will not agree to a lower price. The mechanism described in Section 3 shows how an agent can determine this base price boundary.

In order to separate the essentially different equilibria from among all the competitive equilibria possible in a given market situation, we introduce in Section 4 some classifications of market agents and dwellings by types and groups, respectively. Then we define the "standard" equilibrium. In such an equilibrium, dwellings of the same type are equally priced, consumers from the same group initially occupying dwellings of the same type receive equal utility, and no consumer leaves his/her occupied dwelling for a dwelling of the same type. It is proved that standard equilibria exist and can be described through solutions and dual prices of some linear programming problem.

The partial comparative statics analysis for the studied model is carried out in Section 5. Namely, the consequences of the appearance of an additional (tenantless) dwelling are analyzed. We suggest some methods for calculating the exact boundaries of this dwelling's equilibrium price. The following results assume that the initial equilibrium is disturbed by the appearance of a new dwelling and that some new equilibrium emerges.

The components of any price system in equilibrium for the new situation can be "truncated" to the level established by the initial equilibrium. If the new equilibrium is generated by the market mechanism mentioned above, then only one (equilibrating) chain of agents' movings can be realized; the new dwelling is the last one in this chain. In such an equilibrium, the prices do not exceed initial prices and the closer a dwelling is to the end of the equilibrating chain, the more its price drops. The set of all equilibrating chains coincides with the set of all paths of maximal weight in some graph. This graph allows for the easy construction of a natural equilibrium price system for each equilibrating chain. The new dwelling's price is constant in all such price systems. An example shows that the emergence of an additional dwelling may change the equilibrium without changing either prices or the values of consumers' utility functions.

So, the market will necessarily come to some equilibrium, this equilibrium is effective, and we have the description of all possible equilibria. Based on these results, various approaches to regulating the housing market in order to decrease the number of households lodged in socially unacceptable dwellings are studied in Section 6. Financing housing construction affects the number of tenantless dwellings (regulation of the market from the supply side) and subsidizing market agents affects reservation prices (regulation of the market from the demand side). Some methods are suggested for detecting consumers who need support, selecting the dwelling types for a housing construction program, and determining the value of housing subsidies.

A model for selecting an expedient housing construction program is created. This model rationally distributes the program budget by dwelling type and calculates one of the possible resultant equilibria. A model for selecting an expedient housing subsidies program is also described. This model rationally distributes the program budget among consumers and calculates one of the possible resulting equilibria. By establishing the rules for market agents' access to programmatic dwellings, regulators can influence the rationing scheme and thus "direct" the market to the most desirable equilibrium.

Both of the models discussed can be used for developing profitable housing programs, the earnings from which can partially offset the costs of housing market regulation.

For technical reasons, proofs for all the results are not included in the printed version of the paper. They can be found on EERC's site (www.eerc.ru) or obtained from the author.

1. HOUSING MARKET MODEL

The object of the study is a local (urban, for example) housing market within a short-term period. During this period neither new households emerge nor market agents change their preferences. The model described below presents the market as an exchange economy with money and takes into account the indivisibility of dwellings. It details the model suggested by Quinzii (1984) as applied to a housing market. The models of such type were also by Gale (1960, Chapter 2, § 6), Shapley and Shubic (1972), Shapley and Scarf (1974), Roth and Postlewaite (1977), Kaneko (1983), Demange and Gale (1985).

1.1. Basic concepts and assumptions

The following condition (see Quinzii, 1984, p. 41) is true for the above mentioned models.

Assumption 0. In the considered market, each agent does not initially own more than one indivisible item and has no use for more than one of these items.

The model considered below fulfils Assumption 0. Some of our results are probably true without this condition, but quasilinearity of utility functions is essential. The last conclusion follows from Bevia, Quinzii and Silva (1999), here one can find the review of the works where Assumption 0 was relaxed.

Roth and Postlewaite (1977), and Demange and Gale (1985) consider some abstract indivisible goods; Gale (1960), and Shapley and Shubic (1972) mention dwellings for sale; Kaneko (1983) studies the market of rental dwellings. The likeness of the listed models indicates the possibility of constructing a model covering both sectors of a housing market. Such a model is presented below. The model does not reflect the choice of a dwelling's tenure type, however. We make the following assumption.

Assumption 1. For every dwelling, a tenure type has been determined: a dwelling is either *for sale* (occupied by the owner and/or intended for sale), or *for rent* (rented by a tenant or intended for rent); the dwelling's tenure type is invariable within the period.

Thus we distinguish only two sectors in a housing market. In most countries (and certainly in Russia) there is also a municipal housing sector, with relatively low prices and restricted access (social rent). Incorporation of this sector in the model will be discussed in Section 6.

It follows from Assumption 1 that an owner-occupant can sell his dwelling but cannot rent it out, and a dwelling rented at the beginning of the period should be rented or be tenantless at the end of the period.

We shall use the considered period as a unit of time and call it "year."

The *commodities* are dwellings (all different and indivisible) and money. Let I be the set of all dwellings; $I = I_1 \cup I_2 \cup \{0\}$, where I_1 and I_2 are the sets of dwellings for sale and dwellings for rent, correspondingly; zero symbolizes a *dummy* "dwelling." Each dwelling $i \neq 0$ is owned by some agent $g(i)$. An agent, g, owns dwelling $d(g)$ and occupies dwelling $\delta(g)$ at the beginning of the period; $d(g) = 0$ (respectively, $\delta(g) = 0$), if agent g owns (respectively, occupies) no dwelling in the considered market.

Assumption 0 can be reformulated now as follows.

Assumption 2. The values $d(g)$ and $\delta(g)$ are uniquely defined; if $g \neq h$ and $\delta(g) = \delta(h) = i$, then $i = 0$; if $d(g) \neq 0$ and $\delta(g) \neq 0$, then $d(g) = \delta(g)$.

The last condition means that the owner of a non-dummy dwelling lives either in this dwelling or outside of the considered local market.

Definition. Agent g is a *supplier* if $\delta(g) = 0$ and $d(g) \neq 0$, agent g is a *consumer* otherwise.

Let G be the set of all market agents, $G = G_1 \cup G_2$, where G_1 is the set of all consumers and G_2 is the set of all suppliers.

A supplier owns some dwelling in the considered market but prefers to live outside this market in the beginning of the period. Assuming the agents' preferences to be invariable, let's suppose that each supplier is not going to acquire or to lease a dwelling in the considered market within the year, and he wants to sell or to lease out the dwelling he owns.

Let us formulate some obvious consequences arising from Assumption 1. The dwelling $d(g)$ for $g \in G_2$ is tenantless if $d(g) \in I_1$, and this dwelling can be occupied (rented) by some consumer if $d(g) \in I_2$; a consumer is living in his own or a rented dwelling, or occupies a dummy "dwelling"; a consumer may own only the dwelling he occupies; if $g \in G_1$, then $d(g) \in I_1 \cup \{0\}$. Suppliers "make it possible" to introduce rental dwellings and newly constructed ones into the model without violating Assumption 2.

The set of *consumption bundles* for an agent g is $Z_g = R \times J_g$, where $J_g = I$ for $g \in G_1$ and $J_g = \{0, d(g)\}$ for $g \in G_2$. For $z = (y, i) \in Z_g$ let us put $y(z) = y$ and $j(z) = i$.

By choosing (y, i), an agent defines the money equivalent y of his demand for annual consumption outside the considered market (*non-housing consumption*). By choosing (y, i), consumer g also defines his demand for purchasing or leasing dwelling i (in the case of $i = d(g)$, the agent "buys" $d(g)$ from himself), and the supply of $d(g)$ for sale if $d(g) \notin \{i, 0\}$. For a supplier, g, the choice (y, i) means zero supply if $i = d(g)$ (he leaves the market) and the supply of $d(g)$ for sale or for lease (depending on tenure type) if $i = 0$.

Assumption 3. Each dwelling creates a uniform ad infinitum flow of services (infinitely-lived, non-depreciating assets in a stationary state). Each agent has uniform ad infinitum income flow beyond the considered market. All agents have equal access to perfect capital markets; the interests on deposits and loans are equal and their common value ρ is the same for all agents; all agents have the same (annual) discount factor $(1 + \rho)^{-1}$.

We denote by P the set of all possible systems of prices for dwellings; $P = \{p \in R_+^{|I|} \mid p_0 = 0\}$. Let us fix some price system, $p \in P$. We assume that the components of p are consistent with the dwellings tenure types, namely, p_i is the value of dwelling i if $i \in I_1$, and p_i is the annual rental rate for this dwelling if $i \in I_2$.

Definition. To each vector $p \in P$, we relate the vector of *compatible prices* $c(p) = (c(p_i) \mid i \in I)$ such that $c(p_i) = \rho \cdot p_i$ if $i \in I_1$, $c(p_i) = p_i$ otherwise.

It follows from Assumption 3 that $c(p_i)$ is the price of services created by dwelling i within the year. If agent g leases out dwelling $d(g)$, then he will draw income $c(p_{d(g)})$ in the considered year; if the agent sells this dwelling, then he will gain the stock $p_{d(g)}$, which is equivalent to the flow of annual incomes $c(p_{d(g)})$. In any case, the agent's income will increase by $c(p_{d(g)})$ in the considered year.

Let us denote by q_{gi}^1 and q_{gi}^2, correspondingly, the *non-recurrent* and *recurrent* (annual) *fixed costs* connected to the choice of dwelling $i \in J_g$ by agent g. The "flow equivalent" of the fixed costs is $q_{gi} = q_{gi}^2 + \rho \cdot q_{gi}^1$. As provided by Assumption 3, the annual income of agent g will decrease by $c(p_i) + q_{gi}$ as the result of purchasing or renting dwelling $i \neq d(g)$.

The dwellings prices define the payments among market agents, while fixed costs are payments by market agents to some aggregated *external agent* (persons and institutions beyond the market: intermediaries, government agencies, *etc.*). In fact, choosing a consumption bundle, a market agent demands some services of the external agent, and the price of these services is fixed. Surely, fixed costs involve transaction costs. Also, one can take into account the money value of non-moneyed expenses (*e.g.*, the time and efforts spent for searching for new dwelling, the inconvenience of repairing and moving, *etc*.). For the owner of a non-dummy dwelling, the choice of this dwelling is connected to the costs as well (taxes, maintenance, *etc*.). It is natural to assume that the non-recurrent costs are absent if agent g does not change his place of residence: $q^1_{gi} = 0$ if $i = \delta(g) \neq 0$. If dwelling i does not satisfy consumer g in some aspects (for example, it is too far from his place of work), then compensatory costs may be included in q^2_{gi}. Thus, the value of a dwelling for a consumer consists of some fixed costs determined by factors exogenous to the considered market, and a price created by market competition.

For a given agent, the dummy dwelling is some housing unit (specified by location, size, and so on) outside the considered local market; the agent would like to occupy such a dwelling if he didn't choose any non-dummy dwelling in the considered market. For consumer g, the values q^1_{g0} and q^2_{g0} reflect his expected costs for sufficing his housing needs from beyond the considered market; these costs are determined by housing prices outside the considered market (surely, we suppose these prices are stable). For supplier g, the dummy dwelling may be thought of as the occupied dwelling beyond the considered market; his non-recurrent expenses connected with occupying this dwelling are equal to zero, and his recurrent expenses do not depend on his choice ($d(g)$ or 0). Therefore, assuming that the buyer (renter) bears all fixed costs in the case of selling (renting) a dwelling, we can set $q_{g0} = 0$ whenever $g \in G_2$.

Let us put $\psi_{gd(g)} = b \geq 0$ if supplier g wants to lease out $d(g)$ at least for b money units per year, and $\psi_{gd(g)} = \rho \cdot b \geq 0$ if he wants to sell $d(g)$ at least for b money units; let us also set $\psi_{g0} = 0$.

Definition. Supplier g's *reservation price* for dwelling $i \in J_g$ is $b_{gi} = \psi_{gi} + q_{gi}$.

Thus $b_{g0} = 0$. The definition of $b_{gd(g)}$ for $g \in G_2$ may be justified as follows. If $c(p_{d(g)}) = \psi_{gd(g)}$, then supplier g is indifferent between selling (leasing out) $d(g)$ and leaving the market. Choosing $d(g)$, he refuses annual income $\psi_{gd(g)}$ and bears annual fixed costs $q_{gd(g)}$; thus, the agent is ready to sacrifice at most $b_{gd(g)}$ amount of annual income for this choice.

Definition. Agent g's *utility function* has the form $u_g(z) = y(z) + e_{gj(z)}$, $z \in Z_g$. If $g \in G_1$, then e_{gi} is some moneyed value of the utility, providing to consumer g within the year with the services, created by dwelling i. If $g \in G_2$, then $e_{gi} = b_{gi}$ for $i \in J_g$.

Let us put $Z = \times_{g \in G} Z_g$. We shall call elements of set Z *distributions*. If $z \in Z$, then we put $y(z, g) = y(z_g)$, $j(z, g) = j(z_g)$, $u_g(z) = u_g(z_g)$. Each $z \in Z$ defines the *allocation* (distribution of dwellings) $\zeta(z) = (j(z, g) | g \in G)$.

The purpose of the next section is to describe the agents' budget restrictions and to specify (following the suggestion of R.E. Ericson) the interpretation of quantities e_{gi} for $g \in G_1$.

1.2. Reservation prices

Let w_g be the *full annual income* agent g is earning beyond the considered market. Agent g chooses (y, i) under prices p, solving the optimization problem: $\max\{y + e_{gi} \mid (y, i) \in Z_g\}$ s.t. $y + c(p_i) + q_{gi} - c(p_{d(g)}) \le w_g$.

For $p \in P$, $g \in G$, and $i \in J_g$, let us set $\beta_{gi}(p) = c(p_i) + q_{gi} - c(p_{d(g)}) - q_{gd(g)}$. Suppose agent g has chosen dwelling $i \in J_g$ at prices p; the maximal possible value of his non-housing consumption and that of his utility function we denote by $y_{gi}(p)$ and $v_{gi}(p)$, correspondingly. It is clear that $y_{gi}(p) = w_g - \beta_{gi}(p) - q_{gd(g)}$, and therefore,

$$v_{gi}(p) = y_{gi}(p) + e_{gi} = w_g + e_{gi} + c(p_{d(g)}) - c(p_i) - q_{gi}. \tag{1}$$

Note that $\beta_{gd(g)}(p) = 0$, $y_{gd(g)}(p) = w_g - q_{gd(g)}$, and $v_{gd(g)}(p) = w_g - q_{gd(g)} + e_{gd(g)}$ regardless of prices.

Supplier g can obtain utility $w_g + \psi_{gd(g)}$ if he chooses $d(g)$, or $w_g + c(p_{d(g)})$ if he chooses zero. He will choose zero (will offer $d(g)$ for sale or for rent) if $c(p_{d(g)}) + q_{gd(g)} > b_{gd(g)}$ (*i.e.*, an increment of his annual income and a saving of the fixed costs, in totality, exceed the reservation price), and he will leave the market if $\psi_{gd(g)} > c(p_{d(g)})$.

Let us now consider a consumer's behavior.

Definition. y_g^0 is the minimum annual *non-housing consumption* acceptable for consumer g; clearly, $0 \le y_g^0 \le w_g$.

Assume that the following condition is satisfied:

$$e_{gi} - e_{gd(g)} \le w_g - y_g^0 - q_{gd(g)} \text{ for } i \in I,\ g \in G_1. \tag{2}$$

Consumer g should pay $q_{gd(g)}$ per year for possessing the dwelling $d(g)$, and he cannot allocate more than $w_g - y_g^0$ per year for housing consumption. The increment of housing utility in the case of choice i in comparison with the choice $d(g)$ cannot exceed the increment of the corresponding expenses. This substantiates (2) with $i \neq d(g)$.

Assume that $i = d(g)$; then (2) takes the form

$$w_g - y_g^0 - q_{gd(g)} \geq 0. \tag{3}$$

If $d(g) = 0$ then (3) requires that the sum of the agent's expected costs for sufficing his housing needs beyond the considered market and for non-housing consumption would not exceed his annual income. If (3) is not fulfilled with $d(g) \neq 0$, then the consumer owns a dwelling, which is too expensive for him (and occupies it at the beginning of the period). If he will remain in $d(g)$, then he will not be able to pay $q_{gd(g)}$ in full and will reduce the fixed payments to an acceptable level de facto; eventually he will lose his possessive rights for this dwelling, whereupon $d(g)$ will become zero. Consequently, we assume that all insolvent owners had already lost their dwellings before the beginning of the period. So, (2) is a consistency condition for the model's parameters.

For $(g, i) \in G_1 \times I$, let Δb_{gi} be the greatest value of $\beta_{gi}(p)$ over $p \in P$ subject to

$$v_{gi}(p) \geq v_{gd(g)}(p), \tag{4}$$

$$y_{gi}(p) \geq y_g^0. \tag{5}$$

The value $\beta_{gi}(p) = (c(p_i) + q_{gi}) - (c(p_{d(g)}) + q_{gd(g)})$ is the increment of the agent's annual expenses in the case of choice i, in comparison with choice $d(g)$. Therefore, Δb_{gi} is the maximal additional annual expenses the agent would be willing to bear for the sake of owning/renting dwelling i (if $\Delta b_{gi} < 0$, then the agent will approve choice i under the condition of increasing his non-housing consumption at least by $|\Delta b_{gi}|$ per year as the result of this choice). So, we can interpret Δb_{gi} as the difference in the reservation prices of agent g for the dwellings $i \neq d(g)$ and $d(g)$. The greater this difference, the greater is the utility of dwelling i for agent g. Besides, increasing the reservation price (asking price) for $d(g)$ and expecting nonzero demand for $d(g)$ at this price, the agent can increase his reservation prices (bid prices for purchasing or renting) for other dwellings.

Definition. *Reservation prices* b_{gi} ($i \in I$) of consumer g are the solution to the following set of equations:

$$b_{g0} = q_{g0} \text{ and } b_{gi} - b_{gd(g)} = \Delta b_{gi} \text{ for } i \neq d(g). \tag{6}$$

A reservation price b_{gi} is the maximal annual expenses acceptable for agent g in connection with choice i. In particular, $b_{g0} - b_{gd(g)} = \Delta b_{g0} = \beta_{g0}(p)$ for some $p \in P$, then $b_{gd(g)} = c(p_{d(g)}) + q_{gd(g)}$. That is, $b_{gd(g)}$ includes the missed profit (the "flow equivalent" of the minimum price, which the agent would accept when selling $d(g)$) plus the fixed costs. Note that b_{gi} is agent g's reservation price for the annual service flow from dwelling i; if $i \in I_1$, then the "true" reservation price (bid price of purchasing or selling) is $b_{gi} \cdot \rho^{-1}$.

Theorem 1. If $g \in G_1$, then $b_{gi} = e_{gi} - e_{g0} + q_{g0}$ for all i.

According to Theorem 1, the reservation prices exist and the difference $b_{gi} - e_{gi}$ does not depend on i. Therefore, replacing each e_{gi} by b_{gi} in the utility function u_g, we shall obtain the equivalent utility function. Thus we assume in what follows that $u_g(y, i) = y + b_{gi}$ for all g. Note that conditions (2) remains true after replacing all e_{gk} by b_{gk} since $e_{gi} - e_{gj} = b_{gi} - b_{gj}$ for all i and j according to Theorem 1. Kaneko (1976) had found the necessary and sufficient condition for an individual preference ordering to be represented by a quasilinear utility function. Theorem 1 states that under natural condition (2), an agent's reservation prices for dwellings can be used in his quasilinear utility function as a measure of housing utility.

Remark. If $e_{gi} - e_{g0} \geq 0$ (with the same level of non-housing consumption, dwelling i is not worse than the dummy one to agent g), then $b_{gi} \geq q_{g0} \geq 0$.

Our ultimate purpose is to point out some approaches to the regulation of housing markets. The results of regulatory actions reveal themselves in equilibrium, that is why the following section deals with equilibria for the considered model.

2. EQUILIBRIA

2.1. Walrasian equilibria

In this section, we shall describe all Walrasian equilibria for the considered model by means of some simple linear programming problem and study some properties of equilibria. Existence of Walrasian equilibria was proved by Gale (1984) for a very general model. In some particular cases, Gale (1960) and Kaneko (1983) had found the complete descriptions of all equilibria that makes it possible to calculate some equilibrium for any initial situation. Theorem 2 strengthens the first of these results and is independent of the second one (Kaneko has assumed that all market agents have the same preferences of a general form on the set of dwellings).

Definition. Distribution z is *feasible* if

$$\Sigma_g\, y(z, g) + \Sigma_g\, q_{gj(z,g)} = \Sigma_g\, w_g, \tag{7}$$

$$g \neq h \text{ and } j(z, g) = j(z, h) = i \text{ imply } i = 0. \tag{8}$$

Equality (7) is the condition of market clearing with respect to money (the total of internal payments is zero); (8) means that the demand for each dwelling does not exceed the corresponding supply (the supply of dummy dwellings is not restricted). Denote by FD the set of all feasible distributions.

Definition. A *Walrasian equilibrium* is a collection $(z, p) \in \mathrm{FD} \times P$ such that the following conditions are satisfied:

$$u_g(z) = \max\{u_g(z^1) \mid z^1 = (y, i) \in Z_g,\ y + c(p_i) + q_{gi} - c(p_{d(g)}) \leq w_g\} \text{ for } g \in G, \tag{9}$$

$$\text{if } i \notin \{j(z, g) \mid g \in G\}, \text{ then } p_i = 0. \tag{10}$$

By (9), each agent obtains the maximum utility under the budget restriction, and (10) is the condition of market clearing for dwellings.

The value $\psi_{gi} = b_{gi} - q_{gi}$ we shall call the *pure utility* of dwelling i to agent g. Lemma 1 gives a convenient criterion for verifying condition (9) (Bevia, Quinzii, and Silva (1999) have used the analogous criterion), and Lemma 2 describes some properties of the equilibria.

Lemma 1. Condition (9) is equivalent to $\psi_{gj(z,g)} - \psi_{gi} \geq c(p_{j(z,g)}) - c(p_i)$ for all g and $i \in J_g$.

Lemma 2. Let (z, p) be some equilibrium, and put $\pi = c(p)$.

(a) For all g, $u_g(z) = w_g + \psi_{gj(z,g)} + \pi_{d(g)} - \pi_{j(z,g)}$; if $g \in G_2$, then $u_g(z) = w_g + \max\{\psi_{gd(g)}, \pi_{d(g)}\}$.

(b) If $g \in G_1$, then $y(z, g) \geq y_g^0$.

(c) If $g \in G_2$, then $\pi_{d(g)} \geq \psi_{gd(g)}$ whenever $d(g) \in \{j(z, h) \mid h \in G_1\}$ and $\psi_{gd(g)} \geq \pi_{d(g)}$ otherwise.

It follows from statement (b) of Lemma 2 that all consumers are solvent in equilibrium. Besides, $u_g(z) \geq w_g + \psi_{gd(g)}$ for all g by Lemma 5 and Theorem 3 (see below).

For $\pi \in P$, let us put $r(\pi) = (r(\pi_i) \mid i \in I)$, where $r(\pi_i) = \pi_i \rho^{-1}$ if $\in I_1$ and $r(\pi_i) = \pi_i$ otherwise (reconstruction of the "true" prices).

If dwelling i owned by supplier g is not in demand in equilibrium, then we can reckon that the compatible price of this dwelling is equal to ψ_{gi}, because its owner is willing to "pay" this price. Lemma 3 formalizes this reasoning.

Lemma 3. Assume that $p \in P$, $\pi = c(p)$, and the vectors π^1, p^1 are defined as follows: if $g(i) \in G_2$, then $\pi_i^1 = \max\{\pi_i, \psi_{g(i)i}\}$, π_i otherwise; $p_i^1 = r(\pi_i^1)$ for all $i \in I$. If (z, p) is an equilibrium, then (z, p^1) is an equilibrium too.

Now we shall establish the correspondence between equilibria and the triplets (x, α, π), where x is a basic optimal solution of some linear programming problem and (α, π) is an optimal solution of the dual problem. Suppose S_0 is some sufficiently large integer, $x = (x_{gi} \mid g \in G, i \in J_g)$ is a vector of variables, and let us formulate the linear programming problem T:

$$\max \textstyle\sum_{g,i} \psi_{gi} \cdot x_{gi} \text{ s.t.} \tag{11}$$

$$\textstyle\sum_i x_{gi} = 1,\ g \in G, \tag{12}$$

$$\textstyle\sum_g x_{gi} \leq 1,\ i \in I \setminus \{0\}, \tag{13}$$

$$\textstyle\sum_g x_{g0} \leq S_0, \tag{14}$$

$$x \geq 0. \tag{15}$$

The dual problem T* has the form:

$$\min (\textstyle\sum_g \alpha_g + \sum_i \pi_i) \text{ s.t.} \tag{16}$$

$$\pi \geq 0 \text{ and } \alpha_g + \pi_i \geq \psi_{gi} \text{ for } g \in G,\ i \in J_g, \tag{17}$$

where $\alpha = (\alpha_g \mid g \in G)$ and $\pi = (\pi_i \mid i \in I)$ are the vectors of dual variables corresponding to constraints of problem T.

T is a transportation type problem, thus all its basic feasible solutions (*bfs*) are integral and all *bfs* of problem T* are integral if ψ_{gi} are integers (Papadimitriou and Steiglitz, 1982, section 13.2, Corollary). Thus we can interpret the variables and constraints in problem T as follows: $x_{gi} = 1$ if agent g chooses dwelling i, zero otherwise; each consumer chooses just one dwelling (possibly, a dummy one); each dwelling may be chosen by no more than one consumer; the number of dummy dwellings is not restricted.

Lemma 4 states some properties of duals, in particular it explains why the summand $S_0 \cdot \pi_0$ is absent in objective function (16).

Lemma 4. If (α, π) is the optimal solution of problem T*, then $\alpha \geq 0$ and $\pi \in P$.

For $z \in Z$ and $p \in P$, let us put $x(z) = (x_{gi}(z) \mid g \in G,\ i \in J_g))$, where $x_{gi}(z) = 1$ if $i = j(z, g)$, zero otherwise; put also $\alpha(z, p) = (\alpha_g(z, p) \mid g \in G)$, where $\alpha_g(z, p) = u_g(z) - (w_g + c(p_{d(g)}))$.

Let FS be the set of all integral feasible solutions to problem T (clearly, FS $\neq \varnothing$). For any $x \in$ FS, we define $k(x, g)$, $g \in G$, as follows: $k(x, g) = i$ if $x_{gi} = 1$ ($k(x, g)$ is well defined because each vector $x \in$ FS has a unique non-zero component $x_{gi} = 1$ for any $g \in G$). For $x \in$ FS and $\pi \in P$, let us put $y_g(x, \pi) = w_g + \pi_{d(g)} - \pi_{k(x,g)} - q_{gk(x,g)}$, $z_g(x, \pi) = (y_g(x, \pi), k(x, g))$, $z(x, \pi) = (z_g(x, \pi) \mid g \in G)$.

Theorem 2. If (z, p) is an equilibrium, then $x(z)$ is a basic optimal solution of T and $(\alpha(z,p), c(p))$ is an optimal solution of T*. If x is a basic optimal solution of T and (α, π) is an optimal solution of T*, then $(z(x,\pi), r(\pi))$ is an equilibrium.

Theorem 2 proves the correspondence between equilibria and optimal solutions of problems T and T*. It gives the exact formulation of the result discussed (without reference) by Bevia, Quinzii and Silva (1999, p. 3, 4), adding to this result the convenient description of all equilibria and the efficient way of calculating the equilibria for any market situation. The definitions of $\alpha(z, p)$ and $c(p)$ suggest an interpretation of duals in problem T: $\alpha_g(z, p)$ is the utility obtained by agent g in the equilibrium without his initial income and the revenue from selling the dwelling owned by him; π_i is the compatible price of dwelling i. Theorem 2 implies that each system of equilibrium prices equilibrates any equilibrium allocation of dwellings; this is true in a more general model too (Bevia, Quinzii, and Silva, 1999, the proof of Preposition 2.1).

If consumer g has bought (rented) a new dwelling and has not sold his old one ($d(g) \neq 0$), then, in equilibrium, he owns the "excess" dwelling with zero price. It is the price of $d(g)$ as an asset, because $d(g) \in I_1$. But we have used the flow equivalents of all money amounts in our arguments, thus it follows from Assumption 3 that there would not be demand also for renting this dwelling at a positive price. Let us assume that such a dwelling is taken off the market: the owner reconstructs, or demolishes, or neglects it.

We shall call z an *equilibrium distribution* if (z, p) with some p is an equilibrium. Let E be the set of all equilibrium distributions. Denote by f_0 the optimal value of the objective function in problem T and put $F_0 = \sum_g w_g + f_0$.

Corollary 1. If $z \in \mathrm{FD}$, then $\sum_{g \in G} u_g(z) = \sum_g w_g + \sum_{g,i} \psi_{gi} \cdot x_{gi}(z)$; in particular, if $z \in \mathrm{E}$, then $\sum_{g \in G} u_g(z) = F_0$.

By Corollary 1, the total value of agents' utility functions is the same in all equilibria. Corollary 2 gives the sufficient condition of an equilibrium.

Corollary 2. If $z \in \mathrm{FD}$, $x(z)$ is a basic optimal solution of problem T, and there exists an optimal solution of problem T* such that $u_g(z) = \alpha_g + w_g + \pi_{d(g)}$ for all $g \in G$, then $(z, r(\pi))$ is an equilibrium.

The remainder of this section is devoted to the efficiency analysis of equilibrium distributions (Theorem 3).

An arbitrary set Q such that $\varnothing \neq Q \subseteq G$ will be called a *coalition*. For any coalition Q, we put $D(Q) = \{d(g) \mid g \in Q\} \cup \{0\}$ and $Z(Q) = \times_{g \in Q} Z_g$.

Definition. Distribution $z \in Z(Q)$ is *feasible for coalition* Q if $\{j(z, g) \mid g \in Q\} \subseteq D(Q)$ (members of Q choose dwellings only in $D(Q)$); $\{g, h\} \subseteq Q$, $g \neq h$, and $j(z, g) = j(z, h) = i$ imply $i = 0$; $\sum_{g \in Q} y(z, g) + \sum_{g \in Q} q_{gj(z,g)} = \sum_{g \in Q} w_g$ (agents in Q do not pay to and do not obtain the payments from agents in $G \setminus Q$).

Let $\mathrm{FD}(Q)$ be the set of all distributions feasible for coalition Q. If $z \in \mathrm{FD}$, $z^1 \in \mathrm{FD}(Q)$, and $u_g(z) < u_g(z^1)$ for all $g \in Q$, then we say that coalition Q *blocks* z by means of z^1. The *core* C of the market is the set of all distributions $z \in \mathrm{FD}$ such that no coalition blocks z.

A distribution $z \in \mathrm{FD}$ is *Pareto-maximal* if there is no distribution z^1 in FD such that $u_g(z) \leq u_g(z^1)$ for all $g \in G$ and $u_g(z) < u_g(z^1)$ for some g. Let PM be the set of all Pareto-maximal distributions.

Lemmas 5 and 6, being of autonomous interest, are needed to prove Theorem 3.

Lemma 5. If $z \in C$, then $\sum_{g \in G} u_g(z) = F_0$ and $u_g(z) \geq w_g + \psi_{gd(g)}$ for all $g \in G$.

Lemma 6. Assume that $z \in C$, $x = (x_{gi} \mid g \in G,\ i \in J_g)$, and $c_{gi} = \psi_{gi} - u_g(z) + w_g$ for all $g \in G$, $i \in J_g$. Then the following problem is bounded:

$$\max \textstyle\sum_{g,i} x_{gi} \cdot c_{gi} \text{ s.t.} \tag{18}$$

$$x \geq 0 \text{ and } \textstyle\sum_g x_{gi} - \sum_j x_{g(i)\,j} \leq 0 \text{ for } i \neq 0. \tag{19}$$

Theorem 3. $E = C \subseteq \mathrm{PM}$. If $|G| \geq 2$, then $E \subset \mathrm{PM}$.

Equality $E = C$ means that every distribution lying in the core may be decentralized by means of prices. Quinzii (1984, Theorem 3) has proved $E = C$ with utility functions monotonous and continuous with respect to the quantity of money and under conditions denoted as A.1, A.2, A.3. In our model, A.1 follows from the quasilinearity of utility functions, A.2 is the consequence of (2), and A.3 is equivalent to $e_{gi} - e_{g0} \geq 0$ (see Remark to Theorem 1). We do not use (2) in the proof of Theorem 3. Therefore, if utility functions are quasilinear, then E and C coincide without additional conditions.

Svensson (1983) has studied the market model (let us denote it by SV), which describes the distributions of some number of indivisible goods and some amount of money under weak restrictions on agents' preferences; he has introduced the metric $d((x,i), (y,j)) = |x - y| + |i - j|$. Preference relations created with functions u_g are continuous and locally non-satiated with respect to this metric, whence $E \subseteq \mathrm{PM}$ follows (see Mas-Collel, Whinston and Green, 1995, Proposition 16.C.1); when proving Theorem 3, we give the simple straightforward proof of this fact.

Svensson (1983) has shown that in the model SV, a distribution $z \in \mathrm{PM}$ is an equilibrium one if it satisfies some non-trivial condition; for our model, this condition takes the form

$$u_g(z) > w_g + \psi_{gj(z,g)} \text{ for all } g. \tag{20}$$

But, if (z, p) is some equilibrium and $g \in G_1$, then $u_g(z) = w_g - c(p_{j(z,g)}) + c(p_{d(g)}) + \psi_{gj(z,g)}$ (Lemma 2). Thus (20) implies $c(p_{d(g)}) > c(p_{j(z,g)})$: each consumer chooses in equilibrium a dwelling cheaper than the one initially owned by him. The last statement can be confuted easily (consider, for example, the case of $d(g) = 0$). Consequently, our model differs essentially from the model SV.

2.2. Equilibria with respect to rationing schemes

From Theorem 3 it follows that Walrasian equilibria are the desirable (efficient) states of the market. But is it possible to assert that the market necessarily will come to some equilibrium from any initial situation? The numerosity of equilibria in the considered model is rather the rule than the exception. How does the market "select" one of all possible equilibria? What is the market agents' behavior ensuring transition to some equilibrium? To answer these questions we shall describe some family of rationing schemes; in my opinion, such schemes are inherent in housing markets in the short-run, and we shall study the equilibria with respect to these schemes (under fixed prices). Generally speaking, a rationing scheme is some mechanism peculiar to the specific market, "part of the institutional arrangement of the economy" (Schwödiauer, 1978, p. XXXIII). This mechanism "distributes" the demand and/or supply shortage (if such shortage exists at the current prices) among the market agents.

Equilibria with respect to rationing schemes were introduced by Benassy (1975), Drèze (1975), Younès (1975), Grandmont, Laroque and Younès (1978), Laroque and Polemarchakis (1978). We shall use these definitions for the considered model in the formulation given by Grandmont (1993) with a small modification: the current price of a dwelling may be equal to zero, and agent g's demand for 0 and $d(g)$ cannot be constrained.

The dwellings allocations under fixed prices were analyzed first, seemingly, by Herbert and Stevens (1960). Gustafsson *et al*. (1980, p. 85 – 90) have applied the ideas of this paper to markets of rental dwellings. The authors describe the allocations maximizing "consumer surplus" at given prices. The problem is reduced to some linear programming problem similar to problem T (see Section 2.1). Wiesmeth (1985) has studied "fix-priced equilibria" under the assumption that for each household, the set of acceptable dwellings undistinguished in utility is defined. Khutoretsky (1999) has constructed and reduced to a linear programming problem a model generalizing the models by Gustafsson *et al*. (1980) and Wiesmeth (1985). Fix-priced equilibria studied by Wiesmeth (1985) and Khutoretsky (1999) are the equilibria with respect to some rationing schemes; these schemes are not explicitly described in the mentioned works.

In this section we shall work with some invariable vector $p \in P$ of *fixed prices* for dwellings, taking into account the tenure types (*i.e.*, p_i is the value of dwelling i if $i \in I_1$, and p_i is the annual rental rate for this dwelling if $i \in I_2$).

Definition. Let I^+ be some collection of sets $I^+(g)$ for $g \in G$, satisfying condition $\{0, d(g)\} \subseteq I^+(g) \subseteq J_g$ for all g, and let s^- be some collection of numbers $s^-(g)$ for $g \in G$ such that $s^-(g) \in \{-1, 0\}$ and $s^-(g) = -1$ in the case of $d(g) = 0$. Then the triplet $\sigma = (p, I^+, s^-)$ is called a *market signal*.

A signal describes the constraints of demand and supply perceived by agents. Agent g is not constrained in demand just for the dwellings in $I^+(g)$; an agent's demand for money, dummy dwelling, and his own dwelling cannot be constrained by the definition. Whence it follows that suppliers are not constrained in demand: $I^+(g) = J_g$ for $g \in G_2$. If agent g

is not constrained in the supply of dwelling $d(g)$, then $s^-(g) = -1$, and $s^-(g) = 0$ otherwise; the supply of dummy dwellings is not constrained.

Up to here we have assumed that an agent owning a non-dummy dwelling offers it for sale (rent) if he chooses an other dwelling (Section 1.1). Now we change this agreement as follows.

Assumption 4. When choosing a dwelling other than $d(g)$, agent g offers $d(g)$ for sale or rent if and only if $d(g) \neq 0$ and $s^-(g) = -1$ (the agent's supply is not constrained).

An *allocation* is a vector $\zeta \in \times_{g\in G} J_g$ satisfying condition (8): $g \neq h$ and $\zeta_g = \zeta_h = i$ imply $i = 0$; ζ_g is the dwelling allocated to agent g. Let DA be the set of all allocations.

From Assumption 4 it follows that prices, the choice of dwelling, and constraint $s^-(g)$ uniquely determine the maximal possible non-housing consumption of agent g subject to his budget restriction. Therefore, under fixed prices, we can describe demands only by allocations, assuming that the budget restrictions and condition (7) are fulfilled.

In this section, at fixed prices, we shall consider the utility functions of a general form: $w_g(i)$ for $g \in G$, $i \in J_g$.

Definition. Allocation ζ is *feasible with respect to signal* $\sigma = (p, I^+, s^-)$ if $\zeta \in \times_{g\in G} I^+(g)$ (the agents' choices are matched with the constraints of demand).

Let us denote by DA(σ) the set of all allocations feasible with respect to signal σ. The quantity $U_g(\sigma) = \max\{w_g(i) \,|\, i \in I^+(g)\}$ will be called a *restricted optimum* for agent g.

Definitions. The constraint of demand for $i \in J_g \setminus I^+(g)$ is *binding* to agent $g \in G_1$ if $w_g(i) > U_g(\sigma)$. A signal (p, I^+, s^-) is *orderly* if, for each g, either $s^-(g) = -1$ or the constraint of demand for $d(g)$ is not binding to any agent (only one side of the market may be rationed).

A *trade offer* of agent g is an arbitrary set $\Theta_g \subseteq J_g \setminus \{0, d(g)\}$. A trade offer may be interpreted as some list of dwellings desirable for agent g and different from 0 and $d(g)$. If $d(g) \neq 0$ and $s^-(g) = -1$, then, in agreement with Assumption 4, agent g will offer $d(g)$ for sale or for rent when choosing $i \in \Theta_g$. It is clear that $\Theta_g = \varnothing$ for $g \in G_2$.

Definition. A *rationing scheme* ρ is a rule that for each collection $\Theta = (\Theta_g \mid g \in G)$ of trade offers determines constraints $I^+(\rho, \Theta) = (I^+(\rho, \Theta, g) \mid g \in G)$ and $s^-(\rho, \Theta) = (s^-(\rho, \Theta, g) \mid g \in G)$ such that the signal $(p, I^+(\rho, \Theta), s^-(\rho, \Theta)) = \sigma(\rho, \Theta)$ is orderly and the following conditions hold for all $g \in G$:

the set $\Theta_g \cap I^+(\rho, \Theta, g)$ contains no more than one element, (21)

if $i \in \Theta_g \cap I^+(\rho, \Theta, g)$, then $s^-(\rho, \Theta, g(i)) = -1$, (22)

$\Theta_g \cap I^+(\rho, \Theta, g) \cap \Theta_h \cap I^+(\rho, \Theta, h) = \varnothing$ if $h \neq g$. (23)

A rationing scheme "resolves the conflicts" of trade offers; conditions (21) – (23) ensure the balance of demand and supply in equilibria with respect to a rationing scheme (see below).

Now we shall describe a family $\Re$ of rationing schemes. We hypothesize that the schemes of this kind really act in housing markets in the short-run.

Let $n(g)$, $g \in G_1$, be some *numbering of all consumers*, and let $n_g(i)$ for each $g \in G$ be some *numbering of dwellings* $i \in J_g$ in order of non-increasing values $w_g(i)$. The numbering $n(g)$ specifies an order of consumers' entering the market, and the numbering $n_g(i)$ specifies an order, in which consumer g "examines" the available dwellings (the natural presumption is that the agent examines the better dwellings before the worse ones; the numbering $n_g(i)$ is necessary to arrange the equivalent dwellings).

Definition. $t(g, A) = \operatorname{argmin}\{n_g(i) \mid i \in A\}$ for $g \in G$, $A \subseteq J_g$.

Clearly, $w_g(t(g, A)) = \max\{w_g(i) \mid i \in A\}$.

Numberings $n(g)$ and $n_g(i)$ determine the specific scheme, $\rho \in \Re$. The constraints created by this scheme (see below) for an arbitrary collection of trade offers Θ do not depend on Θ: $I^+(\rho, \Theta) = I^+(\rho)$, $s^-(\rho, \Theta) = s^-(\rho)$.

The constraints-constructing algorithm (we shall name it *ρ-algorithm*) works step by step. At the preliminary step 0, for $g \in G_2$, we set $I^+(\rho, g) = \{0, d(g)\}$ (in accordance with the definition of a signal) and $t(g) = t(g, J_g)$. In other words, each supplier decides whether he will offer his dwelling for sale (rent) at current prices. Also, we define the set of dwellings available to consumers at step 1: $D^1 = (I \setminus \{t(g) \mid g \in G_2\}) \cup \{0\}$ (each supplier g is not constrained in the dwelling choice; if he has chosen $d(g)$, then this dwelling is not available for consumers). If $\delta(h) \notin D^1$ for $h \in G_1$, then we put $\delta(h) = 0$ (consumer h lives in the dwelling owned by some supplier g, $\delta(h) = d(g)$; therefore, h rents $d(g)$ at the beginning of the period; when choosing $d(g)$, the supplier does not extend the terms of the lease and the consumer becomes homeless). We define the set of active consumers and that of vacant dwellings at this step, $A^1 = G_1$ and $F^1 = (D^1 \setminus \{\delta(g) \mid g \in G_1\}) \cup \{0\}$. For each $g \in G_1$, an "initial" subset of the set $I^+(\rho, g)$ should be constructed before the first step; it is $I^+(\rho, g, 0) = \{0, d(g)\}$.

Assume that before step k, the sets D^k, A^k, F^k, and $I^+(\rho, g, k-1)$ are defined and the *initial condition* $D^k = F^k \cup \{\delta(g) | g \in A^k\}$ holds. This condition demands that those dwellings either occupied by active consumers or unoccupied would be available to consumers at step k. The initial condition is fulfilled in the first step and the algorithm ensures it at subsequent steps.

Definition. A sequence $\langle g_1, \ldots, g_n \rangle$ of elements belonging to A^k is *feasible* at step k if $t(g_n, D^k) \in F^k \cup \{\delta(g_1)\}$ and $t(g_s, D^k) = \delta(g_{s+1})$ whenever $1 \le s < n$.

A feasible sequence describes either a *cycle* or *chain* of realizable consumers' moves (Fig. 1).
At any step k, ρ-algorithm calculates the values $t(g, D^k)$ for $g \in A^k$ in order

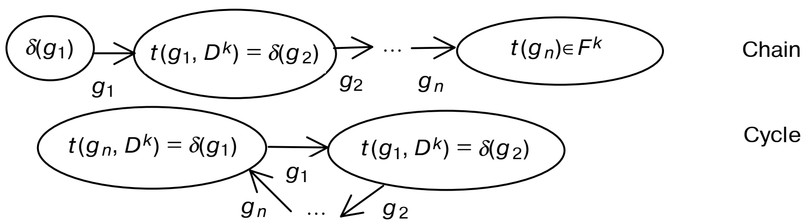

Fig. 1. Sequences feasible at some step k.

of increasing numbers $n(g)$. It is easy to see that the feasible sequences exist if $A^k \ne \varnothing$. Let $\langle g_1, \ldots, g_n \rangle$ be the first such sequence that emerged at step k. Put $t(g_s) = t(g_s, D^k)$ and $I^+(\rho, g_s, k) = I^+(\rho, g_s, k-1) \cup \{t(g_s)\}$, $1 \le s \le n$; now agent g_s is not constrained in demand for dwelling $t(g_s)$. By definition, put: $D^{k+1} = \{0\} \cup (D^k \setminus \{t(g_i) \mid 1 \le i \le n\})$; $F^{k+1} = F^k$ if $t(g_n) = \delta(g_1)$, otherwise $F^{k+1} = (F^k \setminus \{t(g_n)\}) \cup \{0, \delta(g_1)\}$; $A^{k+1} = A^k \setminus \{g_i \mid 1 \le i \le n\}$. Step k is finished, and we turn to step $k+1$.

Obviously, ρ-algorithm will complete its "activity" at some step r such that $A^{r+1} = \varnothing$, and the dwelling $t(g)$ is defined for each $g \in G_1$ after step r. Let us put $I^+(\rho, g) = I^+(\rho, g, r)$ for $g \in G_1$. Put also $s^-(\rho, g) = 0$ if $d(g) \notin \{0\} \cup \{t(g) \mid g \in G\}$, -1 otherwise.

Now $I^+(\rho) = (I^+(\rho, g) \mid g \in G)$ and $s^-(\rho) = (s^-(\rho, g) \mid g \in G)$ are defined, and therefore, $\sigma(\rho) = (\rho, I^+(\rho), s^-(\rho))$ is the *signal created by scheme* ρ. With this $\zeta(\rho) = (t(g) \mid g \in G)$ is the *allocation created by scheme* ρ.

Lemma 7 proves that ρ is really a rationing scheme.

Lemma 7. If $\rho \in \mathfrak{R}$, then $\sigma(\rho)$ is an orderly signal and conditions (21) – (23) are satisfied for any collection of trade offers $\Theta = (\Theta_g \mid g \in G)$.

The schemes belonging to $\mathfrak{R}$ combine the advantages of "queue" and the Gale algorithm (top trading cycle algorithm). The rationing schemes corresponding to the Gale algorithm (see Shapley and Scarf, 1974) create the set of all Walrasian equilibria in particular cases (Roth and Postlewaite, 1977). The idea of combining the "queue" and the Gale algorithm came to me owing the work of Abdulkadiroglu and Sonmez (1998), where these rationing schemes have been investigated under the assumption of strict agents' preferences. The schemes belonging to $\mathfrak{R}$ do not need this assumption.

Definition. Agent g's *effective demand* (see Grandmont, 1993, p. 908) with respect to signal $\sigma = (p, I^+, s^-)$ is a trade offer containing each $i \in J_g \setminus \{d(g), 0\}$ such that $w_g(i) > w_g(j)$ for all $j \in I^+(g) \setminus \{i\}$.

In other words, if a dwelling $i \in J_g \setminus \{d(g), 0\}$ is a unique best choice for agent g subject to (specified with the signal) the constraints of demand for the dwellings other than i but without taking into consideration the constraint of demand for i, then an effective demand should contain i.

Definition. *Equilibrium with respect to rationing scheme* ρ is a vector $\zeta \in \mathrm{DA}(\sigma(\rho))$ such that there exists some collection $\Theta = (\Theta_g \mid g \in G)$ of effective demands with respect to signal $\sigma(\rho)$ satisfying the following conditions:

$$w_g(\zeta_g) = U_g(\sigma(\rho)) \text{ for all } g, \tag{24}$$

$$\zeta_g \in \Theta_g \cup \{d(g), 0\} \text{ for all } g, \tag{25}$$

$$\text{if } s^-(\rho, \Theta, g(i)) = 0, \text{ then } i \notin \{\zeta_g \mid g \neq g(i)\}, \tag{26}$$

$$\text{if } i \notin \{\zeta_g \mid g \in G\} \cup \{0\}, \text{ then } s^-(\rho, \Theta, g(i)) = 0. \tag{27}$$

Conditions (26) and (27) in conjunction with Assumption 4 ensure market clearing under fixed prices.

Theorem 4. If $\rho \in \mathfrak{R}$, then $\zeta(\rho)$ is an equilibrium with respect to scheme ρ.

An equilibrium with respect to any scheme $\rho \in \mathfrak{R}$ exists by Theorem 4; Theorem 5 proves the uniqueness of this equilibrium on the condition that for all $g \in G$,

$$\text{if } d(g) \neq 0 \text{ and } w_g(i) = w_g(d(g)), \text{ then } n_g(i) \geq n_g(d(g)). \tag{28}$$

Condition (28) means that each agent examines his own dwelling before the other equivalent ones. If we consider $t(g, A)$ as the choice made by agent g from set A, then (28) can be interpreted as some voluntariness condition: agent g should not leave $d(g) \neq 0$ for i without gaining some profit (even if $d(g)$ and i are equivalent for him and his move from $d(g)$ to i is profitable for some other agent).

Theorem 5. If $\rho \in \mathfrak{R}$ and condition (28) is true, then $\zeta(\rho)$ is a unique equilibrium with respect to scheme ρ.

Note that in an equilibrium with respect to scheme ρ, a consumer, g, owning a dwelling with a non-zero price can leave it, yet still remaining the owner of this dwelling (if $t(g) \neq d(g)$, $p_{d(g)} \neq 0$, and $s^{-}(g) = -1$); that is impossible in Walrasian equilibria.

Let us prove the efficiency of equilibria with respect to schemes belonging to $\mathfrak{R}$. Let $E(\mathfrak{R}) = \{\zeta(\rho) \mid \rho \in \mathfrak{R}\}$ be the set of all equilibria with respect to such schemes. For $\zeta \in \mathrm{DA}$, put $F(\zeta) = (I \setminus \{\zeta_g \mid g \in G\}) \cup \{0\}$ (the set of all dwellings vacant in allocation ζ).

Definition. $\xi \in \mathrm{DA}$ *weakly dominates* $\zeta \in \mathrm{DA}$ ($\zeta \prec \xi$) if there exists some sequence $\lambda = \langle g(1), \ldots, g(n)\rangle$ of distinct elements of G_1 (*augmentative sequence*) such that

$$\text{if } 1 \leq s < n, \text{ then either } \xi_{g(s)} = \delta(g(s+1)) \neq 0, \text{ or } \xi_{g(s)} = \zeta_{g(s+1)} \neq 0, \tag{29}$$

$$\xi_{g(n)} \in \{\delta(g(1)), \zeta_{g(1)}\} \cup F(\zeta); \text{ if } \xi_{g(n)} \in F(\zeta), \text{ then } n = 1, \tag{30}$$

$$\text{if } 1 \leq s \leq n \text{ and } \xi_{g(s)} \neq \zeta_{g(s)}, \text{ then } w_{g(s)}(\xi_{g(s)}) > w_{g(s)}(\zeta_{g(s)}), \tag{31}$$

$$w_{g(s)}(\xi_{g(s)}) > w_{g(s)}(\zeta_{g(s)}) \text{ for at least one } s. \tag{32}$$

We shall say that allocation $\zeta \in \mathrm{DA}$ lies in the *core under fixed prices p* (fix-price core) and write $\zeta \in \mathrm{FPC}$ if $\zeta \prec \xi$ is not true for any ξ and $w_g(\zeta_g) = \max\{w_g(i) \mid i \in J_g\}$ for all $g \in G_2$.

In other words, allocation ζ lies in the core if the suppliers are not constrained in their choice, and there is no coalition of consumers such that some of its members can become better without changing the other members' choices as revealed by allocation ζ and by choosing only the dwellings either vacant in ζ, or occupied in ζ by the coalition members, or initially occupied by the coalition members.

Theorem 6. $\mathrm{FPC} = E(\mathfrak{R})$.

Note that theorems 4 and 6 do not use condition (28). But for uniqueness of an equilibrium (Theorem 5), some rule is necessary in order to uniquely specify agent g's decision in the case of $w_g(d(g)) = w_g(0) = U_g(\sigma(\rho))$; condition (28) is one of the variants for such a rule.

3. MECHANISM OF EQUILIBRATING THE MARKET

"The theory concerning adjustment processes developed thusfar has not so much value as a description of reality" (Elzen, 1993, p. 5). Nevertheless, the market mechanism described below is plausible enough. Changing prices and allocations, it leads the market to some Walrasian equilibrium through a finite sequence of equilibria belonging to $E(\mathfrak{R})$ (see Section 2.2). That is, the equilibrium appears naturally "as a result of individual actions … but without being designed by any individual" (Hayek, 1955, p. 39)

Existence of such mechanism is especially important in the case of multiplicity of equilibria and gives the theoretical basis for comparative statics analysis and policy implications; see Elzen (1993, p. 4).

We assume that choosing dwellings under fixed prices, the market agents are subject to some rationing scheme $\rho \in \mathfrak{R}$ and attain the equilibrium with respect to this scheme. While searching for a dwelling, as well as in some equilibrium with respect to a rationing scheme, an agent can discover binding constraints of demand and/or supply. He reacts to such constraints by changing the price of his own dwelling. To simplify the further reasoning, we assume that after each change in prices, a new *stage* in the "work" of the mechanism starts. The procedure is finite under the natural assumptions. Stabilization of prices means absence of constraints and, therefore, the emergence of some Walrasian equilibrium with respect to the market situation of the last stage. So, Walrasian equilibria are not only desirable (efficient), but also the natural states of the considered market.

We shall describe the "work" of the mechanism by some *EQ-procedure*. This procedure may be considered as a generalization of MacRae's (1982) "stack-algorithm" (this algorithm, in particular, assumes that, initially, all dwellings are tenantless and all prices are equal to zero). The EQ-procedure acts by stages, and each stage consists of *steps*. A vector of *current compatible prices* $\pi^s \in P$ and the corresponding vector of *current prices* $p^s = r(\pi^s)$ are known at the beginning of any stage s; π^1 is the vector of prices at the beginning of the period, π^s for $s > 1$ is determined at the end of the stage $s-1$. The prices are invariable within a stage and some bargains may be carried out under the current prices at every step.

Let us denote by $G_2(s)$ the set of all suppliers at stage s. Clearly, $G_2(1) = G_2$, while $G_2(s+1)$ is the result of excluding from $G_2(s)$ those suppliers who had sold their dwellings at the stage s. Now we can define the set of all market agents at stage s: $G(s) = G_1 \cup G_2(s)$.

Denote by $\delta(s, g)$ and $d(s, g)$, respectively, the dwelling occupied and the dwelling owned by agent g at the beginning of stage s. If $g \in G_2(s)$, then $d(s, g) = d(g)$. Let us generalize Assumption 2 as follows.

Assumption 5. If $g \in G_1$ and $d(s, g) \neq \delta(s, g)$ then $d(s, g) = 0$.

To justify Assumption 5, let us remember that in a Walrasian equilibrium a consumer may own a non-dummy dwelling that he doesn't live in only if this dwelling is of zero price (Section 2.1). In other words, a consumer lives in the dwelling that he owns until he sells it or becomes convinced that it is impossible to sell this dwelling. In the last case, if he chooses a new dwelling and maintains ownership of the property, then he would bear the current fixed costs connected with this property without any profit; therefore, he would be better off resigning the dwelling (with zero current price) that he owns.

Assumption 6. Each stage has a duration of one "year" and the changes in agents' annual incomes are caused only by their actions in the housing market in the preceding "year."

A consumer's fixed costs connected with the choice of dwelling may depend on the dwelling he occupies, and an agent's reservation prices can depend on his previous actions in the market. These dependencies were irrelevant for us up to now, but now we should take them into account.

We shall use the following notation for $g \in G(s)$: $b_{gi}(s)$, $i \in J_g$, are the agent g's reservation prices at stage s; $b_{gi}(1) = b_{gi}$; $q_{gi}(j) = q^2_{gi}(j) + \rho\, q^1_{gi}(j)$ is the "flow equivalent" of fixed costs connected with the choice of dwelling i by agent g who occupies dwelling j; $q(g, i, s) = q_{gi}(\delta(s,g))$; $\psi_{gi}(s) = b_{gi}(s) - q(g, i, s)$; $w_g(s)$ is the agent g's budget restriction at stage s, $w_g(1) = w_g$. Let us accept the following, obviously natural, assumptions.

Assumption 7. If $g \in G_2(s)$, then $b_{gi}(s) = b_{gi}$, $q(g, i, s) = q_{gi}$, and $w_g(s) = w_g$; if $g \in G_1$, then $q^2_{gi}(j) = q^2_{gi}$ and $q_{g0}(j) = q_{g0}$; $q^1_{gi}(i) = 0$ with $i \neq 0$ for all g.

Assumption 7 means that suppliers' reservation prices, fixed costs, and annual incomes do not depend on s, a consumer's recurrent fixed costs connected with choice i, as well as his minimum expenses for sufficing the housing needs beyond the considered market, are not dependent on the dwelling he occupies, and non-recurrent fixed costs are absent if an agent does not move. The last condition is analogous to the convention we had introduced in Section 1.1 ($q^1_{gi} = 0$ if $i = \delta(g) \neq 0$).

Let $g \in G_1$, $i = \delta(s, g)$, and $j = \delta(s+1, g)$. In agreement with Assumption 6, the values of $w_g(s+1)$ are defined as follows: $w_g(s+1) = w_g(s)$ if $i = j$, $w_g(s+1) = w_g(s) + \pi^s_{d(s,g)} - \pi^s_j - \rho \cdot q^1_{gj}(i)$ if $i \neq j \in I_1$, and $w_g(s+1) = w_g(s) + \pi^s_{d(s,g)} - \rho \cdot q^1_{gj}(i)$ if $i \neq j$ and $j \notin I_1$.

At stage s, an agent g has the utility function $u_{gs}(y, i) = y + b_{gi}(s)$, where y is the value of non-housing consumption within year s, and the agent's budget restriction under prices p is $y + c(p_i) + q(g, i, s) - c(p_{d(s,g)}) \leq w_g(s)$. Thus, if agent g has chosen dwelling i at stage s under prices p, then $y_{gi}(s, p) = w_g(s) + c(p_{d(s,g)}) - c(p_i) - q(g, i, s)$ is his maximum possible non-housing consumption and $v_{gi}(s, p) = y_{gi}(s, p) + b_{gi}(s)$ is the maximum possible value of his utility function. Hence we can assume that agent g evaluates dwelling $i \in J_g$ at stage s under current prices p^s using the following criterion:

$$u(i, g, s) = \psi_{gi}(s) + \pi^s_{d(s,g)} - \pi^s_i\,; \tag{33}$$

this criterion differs from $v_{gi}(s, p^s)$ by the constant $w_g(s)$.

For supplier g, we have $J_g = \{0, d(g)\}$. It follows from Assumption 7 that using criterion (33) he will choose 0 (will offer $d(g)$ for sale or rent) if $\pi^s_{d(g)} > \psi_{gi}$, and he will choose $d(g)$ (will leave the market) if $\psi_{gi} > \pi^s_{d(g)}$.

Analogously to the notation used for describing rationing schemes (Section 2.2), let $n_g(s, i)$ for $g \in G$ be the *numbering of dwellings* in J_g in order of non-increasing values of $u(i, g, s)$; define $t_s(g, A) = \operatorname{argmin}\{n_g(s, i) \mid i \in A\}$ for $g \in G(s)$ and $A \subseteq J_g$. Let also $n(g)$ be the *numbering of market agents*. As in schemes belonging to $\mathfrak{R}$, $n(g)$ determines the order of agents' appearance on the market, and $n_g(s, i)$ is the order, in which agent g examines the dwellings at stage s. We assume that the numbering $n_g(s, i)$ creates the same ordering of any two dwellings j and k at each stage s such that $u(j, g, s) = u(k, g, s)$. (This assumption can be eliminated easily; see comments at the end of this section.) Let us modify condition (28) as follows:

$$\text{if } g \in G_1 \text{ and } u(i, g, s) = u(\delta(s,g), g, s), \text{ then } n_g(s, i) \geq n_g(s, \delta(s,g)) \tag{34}$$

(a consumer "examines" his occupied dwelling, prior to examining other equivalent dwellings).

If $g(i) \in G_2$ and $\pi_i^s < \psi_{g(i)i}$, then the supplier $g(i)$ will not offer i for sale or rent at stage s (he will choose i). Therefore, we introduce a vector of *minimal compatible prices* $\pi^0 = (\pi_i^0 \mid i \in I)$ as follows: $\pi_i^0 = \psi_{g(i)i}$ if $g(i) \in G_2$, zero otherwise. For $s = 1$, we assume that

$$\pi_i^s \geq \pi_i^0 \text{ if } g(i) \in G_2(s) \text{ and } i \in \{\delta(s, g) \mid g \in G_1\} \qquad (35)$$

(if a dwelling owned by a supplier is rented at the beginning of stage 1, then its current price is not less than the minimum one, otherwise the owner would not lease it out). The procedure will make condition (35) true in subsequent stages.

If $k > 1$, then the sets A_s^k (of those consumers who had not chosen the dwelling $\delta(s+1, g)$ before step k), D_s^k (of those dwellings that are the possible choices at step k), and F_s^k (of the dwellings vacant at step k) are determined at step k–1 of stage s. Before step 1 of stage s we define: $A_s^1 = G_1$, $D_s^1 = \{i \in I \mid \pi_i^s \geq \pi_i^0\}$, $F_s^1 = \{0\} \cup (D_s^1 \setminus \{\delta(s, g) \mid g \in G_1\})$. It follows from (35) that $\delta(s, g) \in D_s^1$ for $s = 1$ and $g \in G_1$.

At step k of stage s, each agent $g \in A_s^k$ chooses a dwelling, $t_{sk}(g) \in D_s^k$. Assume that each consumer g remembers his last choice, $L(g)$ (at every step k we set $L(g) = t_{sk}(g)$), and repeats this choice while it remains the best possible choice. The choice rule (the choice of the best accessible dwelling taking into account the numbering $n_g(s, i)$, with *inertia*): if $s > 1$ and $L(g)$ maximizes $u(i, g, s)$ over D_s^k, then $t_{sk}(g) = L(g)$; otherwise $t_{sk}(g) = t_s(g, D_s^k)$. It follows from (34) that $t_{sk}(g) = \delta(s, g)$ if $\delta(s, g)$ maximizes $u(i, g, s)$ over D_s^k.

For a sequence $\lambda = \langle g_1, \ldots, g_n \rangle$, where $g_i \in G_1$ for all i, we shall use the following terminology. The sequence has the *priority* $\max\{n(g_j) \mid 1 \leq j \leq n\}$. It is *feasible* at step k of stage s if $g_j \in A_s^k$ for all j, $t_{sk}(g_j) = \delta(s, g_{j+1})$ whenever $1 \leq j < n$, and $t_{sk}(g_n) \in F_s^k \cup \{\delta(s, g_1)\}$. A feasible sequence is *maximal* if it is not a proper subsequence of another feasible sequence. A feasible sequence is *correct* if, with $i = \delta(s, g_1)$, either $t_{sk}(g_n) = i$, or $\pi_i^s = \pi_i^0$ and $u(i, g_1, s) < u(t_{sk}(g_1), g_1, s)$.

A feasible sequence describes the consumers' moves realizable at step k of stage s; it is a *cycle* if $t_{sk}(g_n) = \delta(s, g_1)$, and it is a *chain* if $t_{sk}(g_n) \in F_s^k$. The definition of a correct sequence requires, above all, that each agent would gain exactly the utility he expected as the result of carrying out the bargains corresponding to this sequence. If λ is a chain, then the preceding condition is true for the owner of dwelling $i = \delta(s, g_1)$ only in the case of $\pi_i^s = \pi_i^0$. In addition, the first participant of a chain should become better off (otherwise he has no need to enter the sequence).

It is easy to see that feasible sequences exist if $A_s^k \neq \varnothing$. The smaller the priority of a sequence, the sooner it will appear if agents make their choices in the ascending order of numbers $n(g)$.

The mechanism "works" in the following way. The correct sequences are being realized (the bargains are carried out at current prices). If correct sequences are absent and there exist feasible (and, therefore, the maximal feasible) sequences, then, in every maximal feasible chain, either the first dwelling is the best for the agent who occupies it, or the price of this dwelling is greater than the minimum one and its owner is constrained in supply. In the first case, agent g_1 chooses $\delta(s, g_1)$; in the second case, the owner of the dwelling $\delta(s, g_1)$ reduces its price. If feasible sequences are absent, then each consumer is a member of some correct sequence and some equilibrium belonging to $E(\mathfrak{R})$ is constructed under current prices. As this takes place, some suppliers probably are constrained in supply and so they will decrease the prices. If no agent perceives constraints, then the market is in Walrasian equilibrium. Let us make the following easy assumption.

Assumption 8. The numbers e_{gi}, $q_{gi}^1(j)$, q_{gi}^2, w_g, y_g^0, and p_i^1 are integers for all i, j, g; ρ is a rational.

Denote by N the denominator of the rational ρ and put $\delta = N^{-1}$. It follows from Assumption 8 that π_i^1 and $\rho \cdot q_{gi}^1(j)$ are rationals. Then $q_{gi}(j)$, ψ_{gi}, π_i^0 are rationals too. It is easy to see that all these numbers are multiples of δ.

Now let us describe the EQ-procedure in a formal way. Assume that

$$D_s^k = F_s^k \cup \{\delta(s, g) \mid g \in A_s^k\} \tag{36}$$

(at step k, a consumer can choose either a vacant dwelling or a dwelling occupied by an agent, g, who had not make his choise $\delta(s+1, g)$ yet; this condition is fulfilled at step 1 of stage 1 and the EQ-procedure ensures it at the subsequent steps and stages). At step k of stage s, either some correct sequence appears or the owner of some dwelling changes the price of this dwelling. As stated above, one should define the sets A_s^1, D_s^1, and F_s^1 before step 1 of stage s. Let us consider three possible situations at step k of stage s.

(1) The correct sequences exist. Let us select the correct sequence $\langle g_1, \ldots, g_n \rangle$ with the least priority and assume that the bargains corresponding to this sequence (consumer g_j moves from $\delta(s, g_j)$ into $t_{sk}(g_j)$, $1 \le j \le n$) are carried out at current prices. By definition, put: $\delta(s+1, g_j) = t_{sk}(g_j)$, $1 \le j \le n$; $D_s^{k+1} = (D_s^k \setminus \{t_{sk}(g_j) \mid 1 \le j \le n\}) \cup \{0\}$; $A_s^{k+1} = A_s^k \setminus \{g_j \mid 1 \le j \le n\}$; $F_s^{k+1} = F_s^k$ if $t_{sk}(g_n) = \delta(s, g_1)$, otherwise $F_s^{k+1} = (F_s^k \setminus \{t_{sk}(g_n)\}) \cup \{0, \delta(s, g_1)\}$. Pass on to step $k+1$ of stage s.

(2) Case 1 does not happen and the feasible sequences exist. Let us select the maximal feasible sequence $\lambda = \langle g_1, \ldots, g_n \rangle$ with the least priority. It follows from definitions of correct and maximal sequence that $t_{sk}(g_n) \ne \delta(s, g_1)$ and $\delta(s, g_1) \notin \{t_{sk}(g) \mid g \in A_s^k\}$ (the first participant of the chain occupies the unsaleable dwelling). Put $j = \delta(s, g_1)$. The following subcases are possible.

(2a) $\pi_j^s > \pi_j^0$. By definition, put: $\pi_j^{s+1} = \pi_j^s - \delta$, $\pi_i^{s+1} = \pi_i^s$ for $i \ne j$; $\delta(s+1, g) = \delta(s, g)$ for all $g \in G_1$ such that $\delta(s+1, g)$ is not defined yet; $r(s) = k$ (hereafter $r(s)$ is the number of the last step of stage s). We proceed to step 1 of stage $s+1$.

(2b) $\pi_j^s = \pi_j^0$. Then $u(j, g_1, s) = u(\delta(s, g_2), g_1, s)$ because λ is not a correct sequence. The dwellings j and $\delta(s, g_2) = t_{sk}(g_1)$ are equivalent for agent g_1; he leaves the chain and remains in j: $\delta(s+1, g_1) = \delta(s, g_1)$. One bargain is carried out. The completion of the case is the same as that in case (1). Pass on to step $k+1$ of stage s.

(3) Cases (1) and (2) do not happen. Then $A_s^k = \varnothing$. Hence it follows that each consumer g had participated in some bargain carried out at some step $m < k$, and $\delta(s+1, g)$ maximizes $u(i, g, s)$ over D_s^m. Note that if a dwelling, i, was occupied at some preceding step and is tenantless at

step k ($i \in F_s^k$), then $\pi_i^s = \pi_i^0$ (see case (2a)). The following subcases are possible.

(3a) $\pi_i^s > \pi_i^0$ for some $i \in F_s^k$ (the owner of such a dwelling is constrained in supply). From Assumption 5 and (2a) it follows that $g(i) \in G_2(s)$ (dwelling i is owned by a supplier). By definition, put $h =$ argmin$\{n(g) \mid g \in G_2(s),\ \pi_{d(g)}^s > \pi_{d(g)}^0\}$. Then we define $\pi_{d(h)}^{s+1} = \pi_{d(h)}^s - \delta$, and $\pi_i^{s+1} = \pi_i^s$ for $i \neq d(h)$. At last we put $r(s) = k$ and pass on to step 1 of stage $s+1$.

(3b) Case (3a) does not happen and there exist some $i \in I$ and $g \in G_1$ such that $i = t_s(g, I)$ and $u(i, g, s) > u(\delta(s+1, g), g, s)$ (consumer g is constrained in demand for dwelling i). Then $i \notin F_s^k$ (a feasible sequence would exist otherwise). Let us select the first such pair (g, i) in the lexicographic ordering created by the numberings $n(g)$ and $n_g(s, i)$. For i belonging to this pair, we set $\pi_i^{s+1} = \pi_i^s + \delta$; prices of other dwellings are invariable. Note that in this case, the price may be increased for a dwelling, i, owned by some supplier, g, who does not offer i for sale or rent at stage s ($d(g) \notin D_s^1$). Put $r(s) = k$ and pass on to step 1 of stage $s+1$.

(3c) Neither case (3a) nor case (3b) happens. Put $r(s) = k$ and finish the procedure.

We shall say that the EQ-procedure at stage s defines allocation $\zeta^s = (j(s, g) \mid g \in G(s))$, where $j(s, g) = \delta(s+1, g)$ if $g \in G_1$, $j(s, g) = 0$ if $g \in G_2(s)$ and $d(g) \in \{\delta(s+1, h) \mid h \in G_1\}$, $j(s, g) = d(g)$ otherwise. The allocation ζ^s and the price vector p^s create a distribution, $z(s) = (z_g(s) \mid g \in G(s))$, with $z_g(s) = (y_{gj(s,g)}(s, p^s), j(s, g))$.

The sets G_1, $G_2(s)$, I_1, I_2, and the vectors $(q_{gi}^1(j))$, (q_{gi}^2) $(\delta(s, g) \mid g \in G_1)$, $(d(s, g) \mid g \in G(s))$, $(b_{gi}(s))$ determine the *market situation* $A(s)$ at the beginning of stage s. If stage s is concluded by case (3), then, obviously, the EQ-procedure operates exactly as ρ-algorithm (Section 2.2) for some rationing scheme $\rho \in \mathfrak{R}$, and $\zeta^s = \zeta(\rho)$ is the equilibrium with respect to scheme ρ under prices p^s in situation $A(s)$. So, if stage s is not the last one, then the EQ-procedure determines the price vector p^{s+1} and the allocation ζ^s; at some stages this allocation is a non-Walrasian equilibrium (belonging to $E(\mathfrak{R})$) under prices p^s.

Theorem 7. If τ is the final stage of the EQ-procedure, then $(z(\tau), p^\tau)$ is a Walrasian equilibrium for situation $A(\tau)$.

Now we shall find out how the consumers' reservation prices change during the EQ-procedure and then we shall prove the finite convergence of this procedure. At stage 1, the reservation prices are determined as described in Section 1.2. Let us formulate the analogous definition of consumer g's reservation prices $b_{gi}(s)$ at stage s > 1 using the utility function $u_{g,s-1}(y, i)$ and the budget restriction $w_g(s)$.

For $g \in G_1$ and $i \neq d(s, g)$, we denote by $\Delta b_{gi}(s)$ the maximal over $p \in P$ value of $\beta_i(s, p) = c(p_i) + q(g, i, s) - c(p_{d(s,g)}) - q(g, d(s,g), s)$ subject to $y_{gi}(s, p) \geq y_g^0$ and $u_{g,s-1}(y_{gi}(s,p), i) \geq u_{g,s-1}(y_{gd(s,g)}(s,p), d(s,g))$.

Definition. If $g \in G_1$, then the reservation prices $b_{gi}(s)$ at step $s > 1$ are a unique solution to the set of equations

$$b_{g0}(s) = q_{g0},\ b_{gi}(s) - b_{gd(s,g)}(s) = \Delta b_{gi}(s) \text{ for } i \neq d(s, g). \tag{37}$$

For $g \in G_2(s)$ and $i \in J_g$, define $b_{gi}(s) = b_{gi}$. If $g \in G_1$, then we denote for short $Q_{gj}(s) = \rho \cdot q^1_{gj}(\delta(s, g))$, $a_{gj}(s) = \pi^s_{d(s,g)} + w_g(s) - Q_{gj}(s) - y^0_g$.

Lemma 8. If $g \in G_1$, $j = \delta(s+1, g)$, and $i \in I$, then $b_{gi}(s+1) = b_{gi}(s)$ in the case when $j \in I_1 \cup \{\delta(s, g)\}$ and $b_{gi}(s+1) = \min\{b_{gi}(s), a_{gj}(s)\}$ otherwise.

Corollary. $\min\{0, b_{gi}(1)\} \leq b_{gi}(s+1) \leq b_{gi}(s)$ for all $g \in G_1$, $i \in I$, $s \geq 1$.

So, the reservation prices do not increase and are bounded below during the EQ-procedure.

Lemma 9. Suppose $g \in G$, $i = j(s, g)$ and case (3) holds at the last step of stage s. Then $b_{gi}(s) - q(g, i, s) \geq \pi^s_i$. If, additionally, $g \in G_1$, then $y_{gi}(s, p^s) \geq y^0_g$.

It follows from Lemma 9 that at every stage of the EQ-procedure each consumer is solvent: he should pay no more than he was going to pay within the year, keeping the acceptable level of non-housing consumption.

Let Al = $\{\zeta^s \mid s \geq 1\}$ and Pr = $\{\pi^s \mid s \geq 1\}$ be the set of all allocations and, respectively, the set of all vectors of current prices created with the EQ-procedure.

Lemma 10. The sets Al and Pr are finite.

It follows from Lemma 10 in particular that the current prices are bounded.

Lemma 11. If the bargains corresponding to a sequence $\lambda = \langle g_1, \ldots, g_n\rangle$ are carried out at stage s of the EQ-procedure and $n > 1$, then $u(\delta(s,g), g, s) < u(\delta(s+1,g), g, s)$ for some $g \in \{g_1, \ldots, g_n\}$.

Lemma 11 states that at least one member of each realized non-trivial sequence of bargains becomes better off. Lemma 12 and Theorem 8 prove the finiteness of the EQ-procedure.

Lemma 12. There exists an integer M such that either the EQ-procedure will be finished before stage $M+1$, or $b_{gi}(s) - b_{gi}(s+1) \geq \delta > 0$ for some i, $g \in G_1$, and $s \leq M$.

Theorem 8. The EQ-procedure is finite under Assumption 8.

So then, the EQ-procedure terminates on some stage τ (Theorem 8) and creates some Walrasian equilibrium for situation $A(\tau)$ (Theorem 7).

In concluding this section, let us make some comments about the EQ-procedure.

1. One may suppose that the numbering of market agents depends on the stage, and the numberings of dwellings are constrained only by condition (34). All results of Section 3 remain true, but, in the proof of Lemma 12, it should be taken into account that the set of all possible numberings is finite. Therefore, the numberings do not restrict the variety of market agents' behavior.

2. In the EQ-procedure, the end of a stage is the signal for changing prices. The market agents cannot, certainly, "catch" this moment in reality, but this is not important. If an owner wants to sell (rent out) his dwelling but cannot do this for quite a long period, then he reduces the price. The reason for increasing the price of a dwelling is simply an offer to buy (rent) this dwelling made to its owner when the dwelling is occupied.

3. Under Assumption 8, all parameters of the EQ-procedure are multiples of δ (one can consider them integers with respect to a suitable unit of measurement). Therefore, if some dwelling was a unique best for an agent before the prices changed at some stage, then it will remain the best (although possibly not a unique best) for him at the next stage. The inertia of choice provides stability of a feasible non-correct sequence until this does not contradict the interests of the participants of this sequence.

4. Why will the price of a dwelling, i, occupied by the first participant of a feasible but non-correct chain decrease at some stage s? The dwelling i is not in demand at the current prices because the chain is a maximal one. Put $g = g(i)$. In the case of $g \in G_1$ (the dwelling is owner-occupied),

the participation in the chain would be profitable for agent g if he sold i for p_i^s. But this is impossible. Agent g is looking for the maximal price providing a non-zero demand for i. Without information on other consumers' reservation prices for i, he "gropes" for this price, decreasing π_i^s by δ. Eventually, at some price such that participation in the chain is profitable for g, the demand for i will possibly arise, the chain will elongate, and g will not have already been the first member of it; or the agent will find out that participation in the chain is not profitable for him at any positive price for i. If $g \in G_2$ (i is a rental dwelling), then, after finding a more suitable dwelling, the occupant informs the owner about terminating a lease. Then supplier g is looking (groping) for the price that maximizes his profit. Eventually, either demand for i will arise under some price not less than ψ_{gi}, or agent g will choose i and leave the market at price ψ_{gi} for i.

5. The utility function value increases for the first participant of the implemented chain and doesn't decrease for all of its other participants. Therefore, the first participant is interested in realizing all bargains and can "redistribute" his gain (for instance, to refund the part of fixed costs connected with each bargain) so that each participant of the chain becomes better off.

6. Using Theorem 2, it is easy to prove that the price vector p^τ equilibrates allocation ζ^τ with respect to situation $A(1)$, if $\psi_{gi}(\tau) = \psi_{gi}$ for all $(g, i) \in G_1 \times I$ and $\psi_{gd(g)} = 0$ for $g \in G_2$. The analogous statement for more general cases has not been proved.

4. AGGREGATING THE MODEL

In this section, we shall construct a linear programming problem that describes all essentially different Walrasian equilibria disregarding inessential details.

Definition. The dwellings i and j are of the same *type* ($i \sim j$) if the following conditions hold: (a) either $\{g(i), g(j)\} \subseteq G_1$, or $\{g(i), g(j)\} \subseteq G_2$ and $\psi_{g(i)i} = \psi_{g(j)j}$, or $i = j = 0$; (b) for any $g \in G_1$, $\delta(g) \notin \{i, j\}$ implies $\psi_{gi} = \psi_{gj}$ and $\delta(g) \in \{i, j\}$ implies $b_{gi} - q_{gi}^2 = b_{gj} - q_{gj}^2$.

We interpret ψ_{gi} as the pure utility of choice i to consumer g, and $b_{gi} - q_{gi}^2$ may be called the pure utility of occupying dwelling i. Thus condition (b) of the preceding definition requires that for dwellings of the same type, the pure utilities of occupying were equal for a consumer occupying one of these dwellings, and the pure utilities of choice were the same for a consumer occupying some other dwelling. If $0 = i \neq j$, then condition (a) of the preceding definition does not hold (because $g(0)$ is not defined); hence, a dummy dwelling and a non-dummy one cannot be of the same type. Note also that $i \sim i$ for all i.

Definition. A Walrasian equilibrium (z, p) is called *normal* if the dwellings of the same type have the same price: $p_i = p_j$ whenever $i \sim j$.

Lemma 13. For any Walrasian equilibrium (z, p), some normal equilibrium (z^0, p^0) exists such that $\zeta(z^0) = \zeta(z)$.

Definition. The consumers g and h are of the same *group* ($g \approx h$) if the following conditions hold: either $\{g, h\} \subseteq G_1$ or $\{g, h\} \subseteq G_2$; $\{g, h\} \subseteq G_1$ implies $w_g = w_h$, $\psi_{gi} = \psi_{hi}$ for $i \notin \{\delta(g), \delta(h)\}$, and $b_{gi} - q_{gi}^2 = b_{hi} - q_{hi}^2$ for $i \in \{\delta(g), \delta(h)\}$; $\{g, h\} \subseteq G_2$ implies $d(g) \sim d(h)$.

Lemma 14. Suppose (z, p) is a normal equilibrium and $g \approx h$. If $\{g, h\} \subseteq G_1$ and $\delta(g) \sim \delta(h)$, then $u_g(z) = u_h(z)$; if $\{g, h\} \subseteq G_2$, then $u_g(z) - u_h(z) = w_g - w_h$.

Definition. A normal equilibrium (z, p) is *standard* if for $g \in G_1$, $j(z, g) \sim \delta(g)$ implies $j(z, g) = \delta(g)$ (a consumer does not change the dwelling he occupies for a dwelling of the same type).

Theorem 9. For any normal equilibrium (z, p), some standard equilibrium (z^0, p) exists such that $j(z^0, g) \sim j(z, g)$ for all $g \in G_1$.

Thus, from an arbitrary equilibrium, one can come to normal equilibrium changing prices of some dwellings and the distribution of money. Then, substituting the dwellings chosen by some agents with equivalent dwellings, one can come to a standard equilibrium. Among all equilibria described with problems T and T*, only the standard ones can appear in a natural way as a result of market competition with full information. Now we shall construct a linear programming problem that describes all standard equilibria in the same way as problem T describes all Walrasian equilibria.

We shall use the following notation: $I(n)$ is the set of all dwellings of type n (dummy dwellings have the type 0); agent g of group h is (n, h)-*agent* if $\delta(g) \in I(n)$ (then suppliers of group h are $(0, h)$-agents); GC is the set of all consumers groups and GS is the set of all suppliers groups. If $h \in$ GS, then all suppliers of group h own dwellings of the same type; denote this type by $\tau(h)$. $G(h, n)$ is the set of all (n, h)-agents, in particular, if $h \in$ GS, then $G(h, 0)$ is the suppliers group h.

The statement of Lemma 15 obviously follows from Assumption 2 and the definitions: the dwellings of type $\tau(h)$ are just the dwellings owned by the suppliers of group h.

Lemma 15. If $h\in$GS, then $I(\tau(h)) = \{d(g) \mid g\in G(h, 0)\}$.

Let U be the set of all triplets (n, k, h) such that $I(k) \cap (\cup_{g\in G(h,n)} J_g) \neq \varnothing$ (some (n, h)-agents can choose some dwellings of type k). For $(n, k, h)\in U$, we define the quantities a_{nkh} as follows: if $h\in$GC, $g\in G(h, n)$, and $i\in I(k)$, then $a_{nkh} = \psi_{gi}$ for $k \neq n$ and $a_{nnh} = \psi_{g\delta(g)}$; if $h\in$GS and $g\in G(h, 0)$, then $a_{00h} = \psi_{g0} = 0$ and $a_{0\tau(h)h} = \psi_{gd(g)}$. Lemma 16 justifies the interpretation of a_{nkh} as the pure utility of dwellings in $I(k)$ to (n, h)-agents.

Lemma 16. The quantities a_{nkh} are well defined.

Let us put $C_n = |I(n)|$, $D_{hn} = |G(h, n)|$. For each triplet $(n, k, h)\in U$, we introduce the variable X_{nkh}: the number of (n, h)-agents choosing dwellings of type k.

The AT problem (aggregative problem T) has the form

$$\max \textstyle\sum_{n,k,h} a_{nkh}\cdot X_{nkh} \text{ s.t.}$$

$$\textstyle\sum_k X_{nkh} = D_{hn}, \tag{38}$$

$$\textstyle\sum_{m,h} X_{mkh} \leq C_k, \tag{39}$$

$$X \geq 0.$$

We assume that C_0 in (39) is sufficiently large. Obviously, if x_{gi} are variables of problem T and $X_{nkh} = \sum_{g\in G(h,n)}\sum_{i\in I(k)} x_{gi}$, then (38) is the result of summing constraints (12) over $g\in G(h, n)$, and (39) is the result of summing constraints (13) over $i\in I(k)$.

The dual problem AT* has the form:

$$\min \left(\textstyle\sum_{h,n} D_{hn}\cdot\gamma_{hn} + \sum_k C_k\cdot\pi_k\right) \text{ s.t. } \gamma_{hn} + \pi_k \geq a_{nkh},\ \pi_k \geq 0.$$

Lemma 17 states that the allocation corresponding to some feasible solution of the AT problem meets the following consequence of Assumption 1: a consumer can choose the dwelling owned by a supplier only if this supplier will choose zero.

Lemma 17. If $h\in$GS and X is a feasible solution of the AT problem, then $\sum_m \sum_{a\in \mathrm{GC}} X_{m\tau(h)a} \leq X_{00h}$.

Theorem 10. Suppose (z, p) is a standard equilibrium, $x = x(z)$, $X_{nkh} = \sum_{g\in G(h,n)}\sum_{i\in I(k)} x_{gi}$ for $(n, k, h)\in U$, $\gamma_{hn} = \alpha_g(z, p)$ for $g\in G(h, n)$, and

$\pi_k = c(p_i)$ for $i \in I(k)$; then X and (γ, π) are optimal solutions of the AT and AT* problems respectively. If X is an integral optimal solution of AT (in particular, a basic one), (γ, π) is an optimal solution of AT*, $\bar{\pi}_i = \pi_k$ for $i \in I(k)$, and $\bar{\pi} = (\bar{\pi}_i \mid i \in I)$, then there exists some basic optimal solution x of T such that $(z(x, \bar{\pi}), r(\bar{\pi}))$ is a standard equilibrium and $X_{nkh} = \sum_{g \in G(h,n)} \sum_{i \in I(k)} x_{gi}$.

So, the AT and AT* problems describe just all standard equilibria. To each integral optimal solution of AT corresponds, generally speaking, some set of equilibrium distributions with minor distinctions: only in dwellings of type k allocation among (n, h)-agents for some triplets (n, k, h).

Consider a dwelling, i, owned by a supplier, g. We can assume that in an equilibrium, the compatible price π_i is not less than ψ_{gi} (Lemma 3). If $\pi_i > \psi_{gi}$, then the dwelling is occupied by some consumer (Lemma 2); if $\pi_i = \psi_{gi}$, then 0 and i are equivalent for agent g and we can suppose he chooses i. Thus the following assumption is reasonable.

Assumption 9. In equilibrium, each supplier chooses his own dwelling if it is not in demand.

Now, in the AT problem, condition (39) with $k = \tau(b)$ may be written as an equality: $\sum_m \sum_{h \in \mathrm{GC}} X_{mkh} + X_{0\tau(b)b} = C_k$. Condition (38) with $n = 0$ and $h = b \in \mathrm{GS}$ has the form $X_{00b} + X_{0\tau(b)b} = D_{b0}$. From here, using $D_{b0} = C_{\tau(b)}$ (Lemma 15), we obtain $\sum_m \sum_{h \in \mathrm{GC}} X_{m\tau(b)b} = X_{00b}$. Put by definition $b_{nkh} = a_{nkh} - a_{0kb}$ if $k = \tau(b)$, a_{nkh} otherwise.

Each variable X_{00h} enters in the objective function of the AT problem with coefficient $a_{00h} = 0$, thus $\sum_{n,k,h} a_{nkh} \cdot X_{nkh} = \sum_{n,k} \sum_{h \in \mathrm{GC}} a_{nkh} \cdot X_{nkh} + \sum_{h \in \mathrm{GS}} a_{0\tau(h)h} \cdot X_{0\tau(h)h} = \sum_{n,k} \sum_{h \in \mathrm{GC}} b_{nkh} \cdot X_{ijh} + \sum_{h \in \mathrm{GS}} a_{0\tau(h)h} \cdot D_{h0}$. Let us eliminate the variables $X_{0\tau(b)b}$ for $b \in \mathrm{GS}$ using the equalities corresponding to conditions (39). We shall obtain the ATC problem: $\max \sum_{n,k} \sum_{h \in \mathrm{GC}} b_{nkh} X_{ijh}$ subject to $\sum_k X_{nkh} = D_{hn}$ for $h \in \mathrm{GC}$, $\sum_m \sum_{h \in \mathrm{GC}} X_{mkh} \le C_k$, $X \ge 0$. The ATC problem does not depend on the variables connected with suppliers; we shall use this property of the problem in Section 6.

The AT and ATC problems are equivalent under Assumption 9. Therefore, the ATC and ATC* problems describe all standard equilibria satisfying Assumption 9. It is easy to see that the dual price of constraint (39) in the AT problem exceeds the corresponding dual price in the ATC problem by ψ_{bk} if $k = \tau(b)$, otherwise these dual prices are equal. In other words, if a dwelling, i, is owned by a supplier, g, then the ATC problem separates the "competitive addition" to the minimum price ψ_{gi} from the compatible price for this dwelling.

5. COMPARATIVE STATICS

A given market situation does not determine a unique equilibrium, as a rule; thus, a comparative statics analysis is complicated in the considered model. The different price systems may equilibrate the same allocation and different allocations may be equilibrated with the same price system. Aggregation (Section 4) does not remove this problem. That is why by now only consequences of the emergence of an additional dwelling have been rather scrupulously analyzed. The results of this section are in line with the analysis of consequences of stimulating housing construction in the book "Жилищная экономика" (1996, p. 130 – 131).

The price of an additional dwelling in some equilibrium for the new situation can be compared only with the price of some dwelling of the same type in the initial equilibrium, thus let us consider the standard equilibria described by the AT problem.

Let $f_0(A)$ be the optimal value of the objective function in the AT problem for market situation A. By Theorem 9 and Corollary 1 from Theorem 2, all Walrasian equilibria for initial situation A have the invariant characteristic: $F_0(A) = \Sigma_g w_g + f_0(A)$; it is the total of all agents' utilities in any equilibrium for situation A. Bevia, Quinzii, and Silva (1999, p. 9) interpret $F_0(A)$ as the social welfare created by the dwellings set I. Suppose that the sets of consumers, initial allocations, and utilities of dwellings coincide in some situations A_1 and A_2, while there is an additional tenantless dwelling, d, of type j in situation A_2. The tenantless in situation A_2 dwelling d may be owned only by some supplier. Thus the definition of a dwelling type implies that all dwellings of type j are owned by suppliers, and there is one additional supplier f in situation A_2. One can consider that in situation A_1 this supplier does not offer dwelling d for sale or rent and, thus, he has utility $w_f + \psi_{fj}$. Then, in an equilibrium for situation A_1, the total utility over all agents participating in situation A_2 is equal to $F_0(A_1) + w_f + \psi_{fj}$. Hence, the "social utility" of dwelling d can be estimated by the difference $\delta = F_0(A_2) - [F_0(A_1) + w_f + \psi_{fj}]$. Denote by $\Pi(A)$ the set of all vectors of compatible prices corresponding to price systems for standard equilibria in situation A. Proposition 3.1 by Bevia, Quinzii, and Silva (1999) is equivalent in our case to Theorem 11.

Theorem 11. $\max\{\pi_j \mid \pi\in\Pi(A_2)\} - \psi_{fj} \le \delta \le \min\{\pi_j \mid \pi\in\Pi(A_1)\} - \psi_{fj}$.

Bevia, Quinzii, and Silva (1999, p. 9) interpret δ as the social utility of a dwelling of type j in situation A_2 as well as the utility of such a dwelling in its "next best use" in situation A_1 ($\delta \geq 0$ by Lemma 3). From Theorem 11 it follows that δ is the lower bound for compatible prices of dwellings of type j in standard equilibria for A_1 and it is the upper bound for compatible prices of such dwellings in standard equilibria for A_2. The additional dwelling of type j will not be in demand if $\min\{\pi_j \mid \pi \in \Pi(A_1)\} < \psi_{fi}$. If $\max\{\pi_j \mid \pi \in \Pi(A_2)\} > \psi_{fj}$, then the dwelling of type j will be in demand. Theorem 10 indicates how to calculate δ. Besides, the minimum (maximal) compatible price for dwellings of type j in any situation A can be found by minimizing (maximizing) π_j subject to $\gamma_{hi} + \pi_k \geq a_{ikh}$, $F(A; \gamma, \pi) \leq f_0(A)$, $\pi \geq 0$ (here $F(A; \gamma, \pi)$ is the objective function of the AT* problem for situation A. So, we can calculate the bounds of prices in standard equilibria and find out whether the additional dwelling of each type will be in demand.

The caused by variation (appearance of a new tenantless dwelling) change in the equilibrium can be observed in more detail if we assume that the market is in equilibrium in the moment of variation. We shall work with the basic (not aggregated) model.

For an initial situation, A, we define the *corresponding allocation* $\zeta(A)$: $\zeta_g(A) = \delta(g)$ if $g \in G_1$, $\zeta_g(A) = 0$ if $g \in G_2$ and $d(g) \in \{\delta(g) \mid g \in G_1\}$, and $\zeta_g(A) = d(g)$ otherwise.

Definition. An allocation, $\zeta = (\zeta_g \mid g \in G)$, is an equilibrium one if there exists some equilibrium (z, p) such that $\zeta = \zeta(z)$ (see Section 1.1).

Assume that the market is equilibrated in situation A_1, *i.e.*, the allocation $\zeta(A_1)$ is an equilibrium one. Then there exists an equilibrium $e_1 = (z^1, p^1)$ such that $\zeta(A_1) = \zeta(z^1)$, in particular, $\delta(g) = j(z^1, g)$ for all $g \in G_1$. Put $\pi^1 = c(p^1)$. Let us denote by F the set of all tenantless dwellings in the situation A_1, $F = \{0\} \cup (I \setminus \{\delta(g) \mid g \in G_1\})$. After adding the tenantless dwelling j to I and including the owner-supplier $g(j)$ of this dwelling into G_2, we shall transform A_1 into the new situation A_2. Let $\lambda = \langle j(1), \ldots, j(n+1) \rangle$ be some sequence of elements from $I \cup \{ j \}$.

Definition. Sequence λ is *feasible* if the following conditions hold: $j(s) \neq j(k)$ whenever $s < k \leq n$, $j(s) \neq 0$ whenever $1 < s \leq n$, $j(n+1) \in F \cup \{j(1), j\}$, and, for each $s \leq n$, there exists an agent, $h(s)$, such that $j(s) = j(z^1, h(s))$ and $j(s+1) \in J_{h(s)}$.

A feasible sequence λ describes the agents' movements possible in situation A_1: agent h_s, who has chosen dwelling $j(s)$ in the initial equilibrium, chooses $j(s+1)$. Let us call such a sequence a *cycle* if $j(n+1) = j(1)$, call it a *chain* otherwise. By the definition, zero can be only the extreme element of a chain. The last participant of a chain chooses either a tenantless dwelling in situation A_1 (a *chain of the first type*) or j (a *chain of the second type*).

Definition. Suppose $\lambda = \langle j(1), \ldots, j(n+1)\rangle$ is a feasible sequence and $j(s) = j(z^1, h(s))$ for $s \le n$; λ is *actualized* in an equilibrium $e = (z, p)$ for situation A_2 if $j(s+1) = j(z, h(s))$ for $s \le n$, and λ is *non-trivial* if $n > 0$.

Given a sequence $\lambda = \langle j(1), \ldots, j(n+1)\rangle$, we put $I(\lambda) = \{j(s) \mid 1 \le s \le n+1\}$ (the set of dwellings involved in the sequence). Let $P(A)$ be the set of all equilibrium price systems for situation A. Put by definition $\delta_i(p) = \pi_i^1 - c(p_i)$ for $p \in P(A_2)$ and $i \in I$ (the difference of compatible prices for dwelling i in equilibria e_1 and (z, p)).

Lemma 18. Let $e = (z, p)$ be an equilibrium for situation A_2 and $\lambda = \langle j(1), \ldots, j(n+1)\rangle$ a sequence actualized in e. (a) If $j(s+1) \ne j$, then $\delta_{j(s)}(p) \le \delta_{j(s+1)}(p)$. (b) If λ is a cycle, then all $\delta_i(p)$ with $i \in I(\lambda)$ are equal. (c) If λ is a chain and $i \in I \cap I(\lambda)$, then $\delta_i(p) \ge 0$. (d) If λ is a chain of the first type and $i \in I(\lambda)$, then $\delta_i(p) = 0$.

Dwelling j is absent in situation A_1 and p_j^1 is not defined. That is why we exclude j in statements (a) and (c) of Lemma 18. Certainly, if the market is in equilibrium e_1, then only one chain of the second type may be actualized as a result of market adjustment after the appearance of dwelling j (the agents cannot become better off owing to other feasible moves). But Lemma 18 compares an arbitrary equilibrium for A_2 with e_1. The non-trivial and non-improving "sequences of distinctions" between the equilibria are possible in this case. By Lemma 18, compatible prices $c(p_i)$ in comparison with prices π_i^1 change by the same value in an actualized cycle, do not change in an actualized chain of the first type, and do not increase in an actualized chain of the second type. In a chain of the second type, the nearer a dwelling is to j, the more it cheapens. Assuming that the actualized chain of the second type leads from the "worse" dwellings to the "better" ones, it may be concluded that a "better" dwelling falls in price no less than "worse" ones.

Let $e = (z, p)$ be some equilibrium for situation A_2. Obviously, each agent $g \in G$ is involved in some actualized sequence (possibly, a trivial one); one of these sequences is maximal by embedding; we denote it by $\lambda_g(z)$.

Lemma 19. If (z, p) is some equilibrium for situation A_2 and $j \notin I(\lambda_g(z))$, then $v_{g\delta(g)}(p) = u_g(z)$.

It follows from Lemma 19 that the allocation obtained by means of "abolition" of all bargains corresponding to cycles and chains of the first type in some equilibrium (z, p) for situation A_2 may be equilibrated by price vector p. The following definition is connected with this reasoning.

Definition. An equilibrium $e = (z, p)$ for situation A_2 is *simple* if the only sequence actualized in e is a chain of the second type.

Let SE be the set of all simple equilibria, and denote by $\lambda(e)$ for $e \in \mathrm{SE}$ a unique sequence (a chain of the second type) actualized in equilibrium e. Put by definition $L = \{\lambda(e) \mid e \in \mathrm{SE}\}$ and $\mathrm{SE}(\lambda) = \{e \in \mathrm{SE} \mid \lambda(e) = \lambda\}$.

All equilibria in $\mathrm{SE}(\lambda)$ create the same allocation, denoted by $\zeta(\lambda)$. Just the allocations $\zeta(\lambda)$ for $\lambda \in L$ may appear when the market, being in equilibrium e_1, adjusts to situation A_2. If $L \neq \varnothing$, then it follows from Theorem 2 that L is the set of all feasible sequences $\langle j(1), \ldots, j(n+1)\rangle$ maximizing $\sum_{s=1}^{n}(\psi_{h(s)j(s+1)} - \psi_{h(s)j(s)})$ subject to $j(s) = j(z^1, h(s))$ for $s \leq n$ and $j(n+1) = j$. A method of constructing all elements of L is described below.

Put $J = I \cup \{j\}$, $H_2 = G_2 \cup \{g(j)\}$ (the sets of dwellings and, respectively, of suppliers in the situation A_2). In situation A_2, an agent g chooses a dwelling from set K_g: $K_g = J$ if $g \in G_1$, $K_g = \{d(g), 0)$ if $g \in H_2$. Put $k(g) = j(z^1, g)$ for $g \in G$. Let us consider a graph $\Gamma = (J, U)$ with the set of vertices J and the set of arcs $U = \{(k(g), i) \mid g \in G, i \in K_g\} \cup \{(0, d(g)) \mid g \in G_2\} \cup \{(0, 0)\}$. A *weight* $c(k, i)$ of the arc (k, i) is defined as follows: $c(k, i) = \psi_{gi} - \psi_{gk}$ if $k = k(g)$, $c(k, i) = \psi_{gi}$ if $(k, i) = (0, d(g))$ with $g \in G_2$, $c(0, 0) = 0$. A *pass* $\mu = \langle i_1, \ldots, i_k\rangle$ in the graph Γ has the *weight* $\Delta(\mu) = \sum_s c(i_s, i_{s+1})$.

Let Γ_1 be the graph obtained from graph Γ by adding the arc $(j, 0)$ with the weight $c(j, 0) = -\psi_{g(j)j}$. As e_1 is an equilibrium, then Theorem 2 implies that each simple and maximal by embedding path of positive weight in Γ_1 corresponds to some chain of the second type (in particular, there are no chains of the first type nor cycles of positive weight in Γ_1). Then there exists a simple path of maximal weight to any vertex of Γ_1. Clearly, $\mathrm{SE} \neq \varnothing$ if and only if there exists a chain λ of the second type (a simple unclosed path in graph Γ with the last vertex j) with the weight $\Delta(\lambda) \geq \psi_{g(j)j}$. All simple paths of maximal weight with the last vertex j can be found using one of the known effective algorithms; see *e.g.* Minieka (1978, Section 3.2). If the weight of these paths is not less than $\psi_{g(j)j}$, then each of them corresponds to some sequence $\lambda \in L$ and thus generates a simple equilibrium.

Let us fix $\lambda = \langle j(1), \ldots, j(n), j(n+1) = j\rangle \in L$. In what follows we assume that $e = (z, p) \in \mathrm{SE}(\lambda)$. Clearly, if a price system, p, is the result of market adjustment to situation A_2 from the initial equilibrium e_1 (*e.g.*, by means of mechanism described in Section 3), then $p \le p^1$. But, generally speaking, in an equilibrium belonging to SE(λ), some prices may be greater than those in e_1. Theorem 12 shows that such prices can be reduced to the level of p^1.

For $p \in P(A_2)$, let us put $\overline{p}_i = \min\{p_i^1, p_i\}$ if $i \in I$, $\overline{p}_j = p_j$, and $\overline{p} = (\overline{p}_i \mid i \in J)$.

Theorem 12. If $(z, p) \in \mathrm{SE}(\lambda)$ and $\bar{z}_g = (y_{gj(z,g)}(\overline{p}), j(z,g))$, then $(\bar{z}, \overline{p}) \in \mathrm{SE}(\lambda)$.

In other words, the allocation $\zeta(\lambda)$ equilibrated by price vector p, may be equilibrated by prices $\overline{p}$ too. Demange and Gale (1985, p. 881), and for a more general model, Bevia, Quinzii, and Silva (1999, Theorem 3.10) have proved that the set of all equilibrium price systems is a lattice. This is true for the considered model too: the set $P(A)$ is a lattice for any situation A. But Theorem 12 does not follow therefrom, because the definition of $\overline{p}$ uses equilibrium prices for two different situations.

Lemma 3 justifies the following assumption.

Assumption 10. If $g(i) \in G_2$, then $\pi_i^1 \ge \psi_{g(i)i}$.

Put $H = G \cup \{g(j)\}$ (the set of all market agents in situation A_2). By modifying the construction suggested by Quinzii (1984, Theorem 3), we shall develop two interesting equilibrium price systems for the allocation $\zeta(\lambda) = (j(\lambda, g) \mid g \in H)$.

In the graph Γ, let μ_i for $i \in J$ be a simple path of maximal weight with the last vertex *i.* Put $\hat{\pi} = (\Delta(\mu_i) \mid i \in J)$, $\hat{p} = r(\hat{\pi})$ (see Section 2.1), $z_g(\lambda, \hat{p}) = (y_{gj(\lambda,g)}(\hat{p}), j(\lambda,g))$, $z(\lambda, \hat{p}) = (z_g(\lambda, \hat{p}) \mid g \in H)$, and $e(\lambda, \hat{p}) = (z(\lambda, \hat{p}), \hat{p})$.

Theorem 13. $e(\lambda, \hat{p}) \in \mathrm{SE}(\lambda)$ and $\hat{p}_i \le p_i^1$ for $i \in I$.

Corollary 1. If $i \in I$, then $\hat{p}_i \le \min\{p_i \mid p \in P(A_1)\}$.

Corollary 2. If $(z, p) \in \mathrm{SE}(\lambda)$, then $p_i \le \hat{p}_i$ for $i \in I(\lambda)$.

Corollary 3. If $i \in I$, $k \in I(\mu_i)$, and $\hat{p}_k < p_k^1$, then $\hat{p}_i < p_i^1$.

Notice that $\hat{p}_i < p_i^1$ (Theorem 13) and Corollary 3 with $i \in I(\lambda)$ are covered by Lemma 18, because this lemma is true for any price vector in $P(A_2)$.

So, $\hat{p}$ is an equilibrium price system for situation A_2. The vector $\hat{p}$ determines the upper bounds of prices for dwellings involved in the sequence λ; these bounds and the prices of other dwellings in equilibrium $e(\lambda, \hat{p})$ do not exceed the minimum equilibrium prices for situation A_1 (Corollary 1). If the price of a dwelling, k, decreases in equilibrium $e(\lambda, \hat{p})$, then the prices of all dwellings i such that in graph Γ, the path of maximal weight with the last vertex i comes through k, decrease too (Corollary 3). On the "main" path λ, closer a dwelling is to j, the more it falls in price (Lemma 18).

It is easy to see that in equilibrium $e(\lambda, \hat{p})$, the prices of dwellings not involved in sequence λ should not necessarily be the maximal equilibrium prices; and what is more, if $I(\mu_i) \cap I(\lambda) = \varnothing$, then $\hat{p}_i$ is the minimum equilibrium price for dwelling i. We shall construct the system of minimum equilibrium prices for arbitrary situation A. The notation for parameters of situation A will be the same as that for situation A_1, but we do not assume that the market is equilibrated in situation A.

Let $\zeta = (\zeta(g) \mid g \in G)$ be some equilibrium allocation for situation A. Consider a graph $\Gamma(\zeta) = (I, U)$ with the set of vertices I and the set of arcs $U = \{(\zeta(g), i) \mid g \in G, i \in J_g\} \cup \{(0, d(g)) \mid g \in G_2\} \cup \{(0, 0)\}$. Let us assign a *weight* $c(k, i)$ to the arc (k, i) as follows: $c(k, i) = \psi_{gi} - \psi_{gk}$ if $k = \zeta(g)$, $c(k, i) = \psi_{gk}$ if $(k, i) = (0, d(g))$ with $g \in G_2$, and $c(0, 0) = 0$.

Theorem 2 implies that there are no cycles of positive weight in $\Gamma(\zeta)$. Thus, for each vertex i, there are paths of maximal weight with the last vertex i, and there is a simple path among these paths; denote it by μ_i. Put $\tilde{\pi} = (\Delta(\mu_i) \mid i \in J)$, $\tilde{p} = r(\tilde{\pi})$ (see Section 2.1), $z_g(\tilde{p}) = (y_{g\zeta(g)}(\tilde{p}), \zeta(g))$, $z(\tilde{p}) = (z_g(\tilde{p}) \mid g \in G)$, and, finally, $e(\tilde{p}) = (z(\tilde{p}), \tilde{p})$.

Theorem 14. In situation A, $e(\tilde{p})$ is an equilibrium. If $p \in P(A)$ and $c(p_{d(g)}) \geq \psi_{gd(g)}$ for all $g \in G_2$, then $\tilde{p} \leq p$.

So, $\tilde{p}$ is the system of the minimum equilibrium prices in the class of equilibria (z, p) satisfying condition $c(p_{d(g)}) \geq \psi_{gd(g)}$ for all $g \in G_2$. This condition is not restrictive (see Lemma 3). Applying Theorem 14 to allocation $\zeta(\lambda)$, one can find the system of the minimum equilibrium prices for situation A_2.

Lemma 20. Suppose $e = (z, p)$ is some equilibrium for situation A_2, $p \leq p^1$, and $g \in G_1$; then $u_g(z^1) \leq u_g(z)$.

It follows from Theorem 12 that any equilibrium allocation for situation A_2 can be equilibrated by prices not exceeding p^1. According to Lemma 20, no consumer becomes worse off in comparison with e_1 at such prices (the statement of this lemma is trivial for a consumer g who owns no dwelling, but it is not quite obvious if $d(g) \neq 0$). The following example shows that emergence of an additional dwelling may change the equilibrium without changing prices and values of consumers' utility functions. We shall consider the AT problem and use the corresponding notation (Section 4).

Assume, for example, that in some aggregated situation A there is a sequence of triplets $(i(s), j(s), h(s))$, $1 \leq s \leq n$, satisfying the conditions $D_{h(s)i(s)} > 0$ and $D_{h(s)i(s+1)} > 0$ (there are dwellings of both types $i(s)$ and $i(s+1)$ occupied by agents of group $h(s)$); $\Delta = \sum_{s=1}^{n-1}(a_{i(s)j(s+1)h(s)} - a_{i(s)i(s)h(s)}) > 0$; $j(s) = i(s+1)$ for $s < n$. Let us select a sequence λ maximizing Δ among those satisfying the above conditions. For simplicity, we set $\psi_{gd(g)} = 0$ for $g \in G_2$ (each supplier is willing to sell or to lease out his dwelling at any price). Suppose the market is in equilibrium $e_1 = (z^1, p)$ in situation A. Assume that $\pi_{i(1)} = 0$ and put $\pi = c(p)$. Denote by AT(A) and AT*(A) the AT and AT* problems for situation A, and let x and (γ, π) be some optimal solutions of these problems. It may be considered that $x_{iih} = D_{hi}$ (for all i, h) and $x_{ijh} = 0$ if $i \neq j$ (there are no moves). Then

$$\gamma_{i(s)h(s)} + \pi_{i(s)} = a_{i(s)i(s)h(s)},\ \gamma_{i(s)h(s)} + \pi_{i(s+1)} \geq a_{i(s)i(s+1)h(s)},$$
$$\gamma_{i(s+1)h(s)} + \pi_{i(s+1)} = a_{i(s)i(s+1)h(s)},\ \gamma_{i(s+1)h(s)} + \pi_{i(s)} \geq a_{i(s)i(s)h(s)}.$$

Therefrom one can easily obtain $\gamma_{i(s)h(s)} = \gamma_{i(s+1)h(s)}$. Thus

$$a_{i(s)i(s+1)h(s)} - a_{i(s)i(s)h(s)} = \\ = (\gamma_{i(s+1)h(s)} + \pi_{i(s+1)}) - (\gamma_{i(s)h(s)} + \pi_{i(s)}) = \pi_{i(s+1)} - \pi_{i(s)}$$

and

$$\Delta = \sum_{s=1}^{n-1}(\pi_{i(s+1)} - \pi_{i(s)}) = \pi_{i(n)} - \pi_{i(1)} = \pi_{i(n)} \text{ (since } \pi_{i(1)} = 0).$$

The appearance of a tenantless dwelling of type $i(n)$ creates a new situation B such that parameter $C_{i(n)}$ in B is greater by one than that in A. Let us define a vector y: $y_{i(s)i(s)h(s)} = x_{i(s)i(s)h(s)} - 1$, $y_{i(s)i(s+1)h(s)} = 1$, and $y_{ijh} = x_{ijh}$ for all other triplets (i, j, h) (one $(i(s), h(s))$-agent moves to a dwelling of type $i(s+1)$). Denote by AT(B) and AT*(B) the AT and AT* problems for situation B. It is easy to check that y and (γ, π) are feasible solutions of AT(B) and AT*(B) respectively. Let f, g^A, and g^B be the objective functions of the AT(A), AT*(A), and AT*(B) problems correspondingly. Then $f(x) = g^A(\gamma, \pi)$ and $\Delta = f(y) - f(x) = \pi_{i(n)} = g^B(\gamma, \pi) - g^A(\gamma, \pi)$. Thus $f(y) = g^B(\gamma, \pi)$, and, consequently, y and (γ, π) are optimal solutions of the AT(B) and AT*(B) problems correspondingly. By Theorem 10, the allocation detected by vector y corresponds to some equilibrium $e_2 = (z^2, p)$ for situation B. The value of an (i, h)-agent's utility function is $w_h + \gamma_{ih} + \pi_{d(h)}$ in both equilibria e_1 and e_2.

The EQ-procedure (Section 3) explains why the moves can happen that, seemingly, gave no benefits to consumers.

Assume that an additional dwelling j of type $i(n)$ enters the market at a compatible price $\pi_{i(n)}$. At stage 1 of the EQ-procedure, each consumer will choose the dwelling he occupies. At the last step, case (3a) will happen, hence the price for j will become $\pi_{i(n)} - \delta$. At prices p, for $(i(s), h(s))$-agents, the dwellings of types $i(s)$ and $i(s+1)$ are equivalent (as $a_{i(s)i(s+1)h(s)} - a_{i(s)i(s)h(s)} = \pi_{i(s+1)} - \pi_{i(s)}$), and, under the same conditions, such dwellings are not worse than the others. That is why dwelling j will be the best for $(i(n-1), h(n-1))$-agents at stage 2. With suitable numberings of the agents and dwellings, a chain $\langle g_{n-1}\rangle$ with $g_{n-1} \in G(h(n-1), i(n-1))$ will appear. This chain will be actualized if $\pi_{i(n-1)} = 0$.

If the chain $\langle g_{n-1}\rangle$ is not correct, then the price of some dwelling j_{n-1} of type $i(n-1)$ occupied by agent g_{n-1} will decrease by δ according to case (2a), and this dwelling will become the best for $(i(n-2), h(n-2))$-agents, and so on. Note that the chain will not "fall apart": once the agent g_{n-s} has chosen dwelling j_{n-s}, he continues choosing it at the subsequent steps and stages (because $L(g_{n-s}) = j_{n-s}$). It follows from $p_{i(1)} = 0$ that with suitable numberings of the agents and dwellings the bargains corresponding to sequence λ will be actualized at some stage. Then the equilibrium with respect to some scheme $\rho \in \Re$ will appear and case (3b) will happen (since $D_{h(s)i(s+1)} > 0$). As a result, the prices of all dwellings involved in chain λ will increase by δ and then return to the initial level. A new equilibrium will appear. On further increasing the number of tenantless dwellings of type $i(n)$, the prices will decrease only after all $(i(s), h(s))$-agents occupy the dwellings of type $i(s+1)$ (for some s).

6. APPLICATIONS TO THE DEVELOPMENT OF HOUSING POLICY

The literature devoted to housing policy and the ways of regulating the housing markets is quite extensive (*e.g.*, Cullingworth, 1967; Donnison, 1967; Yeates and Garner, 1976; Rothenberg and Galster, 1991; Бессонова, 1993, "Жилищная политика …", 1998; "Жилищная экономика", 1996). However the models taking into consideration the indivisibility of dwellings, to my knowledge, haven't been applied to the problem of housing market regulation.

State regulation of the housing market is necessary even in advanced countries, and it is all the more important in Russia. Yeates and Garner (1976, p. 412) wrote: "There is now fairly general acceptance ... of the need for public intervention in the private housing market, stemming either from the fact that people find the housing conditions of the poor morally shocking in an affluent society or because they fear that such conditions are a threat to the very existence of that society." Moreover, the authors of the book "Жилищная политика ..." (1998, p. 46) point out that "in the theory of local public administrating, there is no scientifically established methodology for developing a local housing policy." The approaches to developing local housing programs suggested below are not universal but may be useful in many cases.

6.1. Statement of the problem

Rothenberg and Galster (1991, p. 293) have formulated the goal of housing policy in a most general view as "improvement of a housing market behavior." Housing policy is usually aimed at providing sufficient housing supply, supporting socially acceptable standards, and financially supporting families unable to pay market prices. Donnison (1967) recognizes the following types of "interrelations" between the state and housing market.

1. *Support of free market* (Turkey, Greece). The state makes efforts to increase housing construction regardless of the types of the dwellings being built. For that purpose, the authorities stimulate private building businesses and conduct institutional reforms enabling them to channel more means to housing construction.

2. *Social housing programs* together with a free market (Great Britain, Switzerland). Government housing programs serve specific groups of the population and are supposed to have a provisional impact on the free market. The state doesn't take responsibility for the housing conditions of all its citizens.

3. *All-embracing housing policy* (Sweden, France). The state establishes and controls the housing market to such an extent that the responsibility for the housing conditions of most of the population falls on to the state. Housing policy is combined with other social policies to achieve national goals.

In the former USSR, the third type of housing policy has been actualized before the first and second. The state social guaranties can hardly be reduced to an acceptable level now because of political reasons and because of the low incomes of most of the population. On the other hand, the budget provides no possibility of allocating sufficient funds for social housing programs. That is why the current housing policy of the Russian authorities combines elements of all three approaches. In the Law of the Russian Federation "On foundations of the federal housing policy" (in the version of the Federal laws of 12.01.96 № 9-ФЗ, of 21.04.97 № 68-ФЗ, of 10.02.99 № 29-ФЗ, of 17.06.99 № 113-ФЗ), the goals and instruments of the housing policy are formulated as follows.

The goal of federal housing policy is: to provide social guarantees in the realm of citizen's housing rights; to implement the construction of government, municipal, and private housing facilities; to create conditions for attracting non-budgetary financial sources; to stimulate private ownership; to develop competition in the construction, maintenance, and reparation of housing facilities. The state helps its citizens who are not provided with dwellings that meet living standards by constructing state and municipal housing facilities and using a system of compensation (subsidies) and privileges to pay for housing construction, maintenance, and reparation. The state authorities and local officials should ensure that conditions for renting dwellings within the limits of the social standards are available for citizens, that the possibility of credit support and tax privileges for acquiring or renting a housing is available; they also should carry out housing construction within the budgeted costs in order to provide citizens with dwellings via renting, or purchasing. The state authorities and local administration should provide citizens with compensation (subsidies) in order to make possible the payments for housing within the social standards for living space and for public utilities consumption, taking account of a household's total income.

In fine, the law provides the authorities with the possibility to regulate the housing market and makes them responsible for this regulation. But the authorities cannot influence the market to a sufficient extent due to budgeted deficit at all levels. Mortgage lending and other long-term regulatory measures are being implemented very slowly for obvious reasons: overall instability, low incomes of most of the population, the underdeveloped banking system, difficult access to legal services, and so on.

We leave aside the wide and interesting field of investigation concerning the analysis and comparative study of various methods of housing market regulation. We shall only describe some models, which may be useful tools for such investigation. Taking into account the specific features of the model presented in Section 1.1, we shall speak only about a short-term housing policy of city authorities. In terms of the AT problem (Section 4), this task may be stated as follows.

Assume that before the beginning of the considered period the policy developer (Regulator) forecasts a market situation, A_0, at the beginning of the period, solves the corresponding AT and AT* problems, and determines the *forecasted equilibrium* e_0. Probably, in this equilibrium, consumers of some groups occupy socially unacceptable dwellings (too bad, or too expensive as compared with incomes, or located in unfavorable regions, *etc*.) and some suppliers take their "good" dwellings off the market because there is no demand for these dwellings. It is necessary to change the parameters of the initial situation so that a resultant equilibrium would be free from these disadvantages whenever possible. It is clear that regulation of a market by means of changing some parameters of the initial situation requires expenses. The limitation of these expenses should be taken into account. In Section 6.4, we shall discuss the possible sources of means for covering such expenses. Now let us suppose that expenses are restricted by an amount K.

In the AT problem, only two groups of parameters are partially controlled: components of vectors

$$C = (C_i \mid i \in I) \text{ and } a = (a_{ijh} \mid (i, j, h) \in I^2 \times H),$$

where I and H are the sets of dwellings types and consumers groups, correspondingly. By increasing C_i (the number of dwellings of type i), we influence the market from the supply side. To increase a_{ijh} (the "pure" utility of a dwelling of type j to a representative (i, h)-agent g), one can either subsidize this agent under the condition that he moves into a dwelling of type j (owing to this b_{gj} increases) or grant him some privileges in fixed payments (owing to this q_{gj} decreases). Thus a variation of a_{ijh} is an impact on the market from the demand side.

The Regulator's problem is to work out a program of housing construction (to distribute the amount K of money among dwellings types) or a program of housing subsidies (to distribute the amount K among market agents). Each regulatory program creates a new market situation A_{reg}. The Regulator's goal is to select a program as to maximize the number of those consumers who occupy acceptable dwellings in equilibria for this situation.

Let $D(h)$ be the set of types of those dwellings that are acceptable for consumers of group h. Then $W = \{(i, j, h) \in U \mid j \in D(h)\}$ is the set of expedient moves (the set U was introduced in Section 4). Consider a linear programming problem

$$\max \textstyle\sum_{(i,j,h)\in W} x_{ijh} \text{ s.t.} \tag{40}$$

$$\textstyle\sum_{i,h} x_{ijh} \le C_j, \tag{41}$$

$$\textstyle\sum_j x_{ijh} = D_{hi},\ f(x) \ge f_0,\ x \ge 0, \tag{42}$$

where $f(x)$ is the objective function of the AT(A_0) problem and f_0 is the best value of $f(x)$ in this problem.

The polytope of the problem (40) – (42) is an optimal face of the polytope corresponding to the AT(A_0) problem . Therefore, any basic optimal solution of the problem (40) – (42) is a basic optimal solution of AT(A_0), and so it determines some equilibrium $e_{\max}$ with the greatest number of the agents occupying acceptable dwellings. Similarly we can find the worse (with respect to attainment of the goal) equilibrium $e_{\min}$.

6.2. Market regulation from the supply side

The local authorities can stimulate housing construction and/or finance it directly. The incentives can be aimed at either decreasing a producer's expenses (tax privileges) or attracting investments (insuring mortgage credits). Such indirect stimulation provides long-term effects, which, unfortunately, cannot be taken into consideration in the framework of the studied model. Therefore, we shall assume that the authorities develop some housing construction program and intend to invest sum K in it.

Let N be the set of all pairs (i, h) such that some (i, h)-agents occupy the socially unacceptable dwellings in the forecasted equilibrium e_0. Then the set of *target agents* (the consumers who should be helped by the future program) has the form $\cup_{(i,h)\in N} G(h, i)$.

As we saw in Section 5, each *programmatic dwelling* gives rise to some chain of moves. This chain is useful for achieving the Regulator's objective if it involves a move of a target consumer into an acceptable dwelling. The useful chain may incorporate the moves of non-target agents if these moves vacate some dwellings acceptable for target agents. It is easy to find out what types of dwellings are acceptable to target agents by social standards. However, constructing dwellings of other types may be also convenient. If one maximizes and minimizes the function $\sum_{(i,h)\in N}\sum_{j\in D(h)} x_{ijh}$ under restrictions (41) and (42) (Section 6.1) and then compares the obtained equilibria, then he can discover the dwellings target and non-target agents are competing for. Such dwellings are worthy of being built. The dual prices of constraints (41) indicate the relative usefulness of dwellings types for attaining the program's goal in the best equilibrium: it is necessary to construct the dwellings of the types having positive dual prices. However this reasoning, first, is true only in the area of constant dual prices (this area can be easily described; see, *e.g.*, Гдалевич, 1975, p. 35), and second, it does not indicate how many dwellings of each type should be built.

On the other hand, for any given housing construction program one can solve the corresponding AT(A_{reg}) problem, find the best value, f_0, of its objective function, and, using the problem analogous to the problem (40) – (42), determine e^{reg}_{max} and e^{reg}_{min}: the best and the worst equilibria, from the Regulator's point of view, for situation A_{reg}. Comparing these equilibria, one can evaluate the considered program. So, by varying components of vector C, one can empirically select a rational housing construction program.

Let us now go to the description of a problem, which finds a convenient distribution of sum K among dwellings types for an arbitrary (forecasted) initial situation.

Let I be the set of dwellings types in the forecasted situation A_0 and IP the set of dwellings types acceptable for construction within the scope of a program in the Regulator's opinion (*programmatic types*). Then $J = I \cup IP$ is the set of dwellings types in situation A_{reg} resulting from the program. Note that in situation A_{reg}, the number of dwellings of type $i \in IP \setminus I$ may be equal to zero (if it appears that construction of such dwellings is not expedient).

Regulator plays the role of an owner-supplier (hence all dwellings of programmatic types should be owned by suppliers). Regulator specifies the reservation price of a programmatic dwelling; this price may be less than, or equal to, or greater than the flow equivalent of construction costs.

The arguments in Sections 2.2 and 4 show that the composition of a chain created by a programmatic dwelling depends on the order of agents' appearance on the market. If access to a programmatic dwelling is free, then a target agent may be beaten in competition; even a competitive agent may be left without this dwelling if another agent is ahead of him. Then the program would provide more rich or more active agents with dwellings. Therefrom the natural constraint appears: only target agents have access to the dwellings built at the budgeted cost (this doesn't exclude the stimulation of constructing dwellings intended for other consumers). The generally received rule is the following: access to non-programmatic dwellings is free; a target agent has access to a programmatic dwelling if this dwelling matches the socially guaranteed minimum for this agent.

The rules of access to programmatic dwellings may differ from those for existing dwellings of the same type; if so, let us integrate the programmatic dwellings into some new type. Note that the ban on (i, h)-agents' access to dwellings of type j can be reflected by two equivalent ways in the ATC problem (Section 4). First, one can simply not introduce the variable X_{ijh}. Second, one can "suppress" this variable with a small value of b_{ijh} (for instance, you can suppose that an (i, h)-agent g should offer a bribe if he wants to violate the access rule so that the fixed payment q_{gj} is very large). Therefore, the ATC problem as well generates the standard equilibria under access restrictions for some agents (rich, for example) to some dwellings (belonging to a social housing resource, for example). Let GC be the set of consumers groups in situation A_0 (it is the same in situation A_{reg}). We describe access rules by a set V of triplets $(i, j, h)\in J^2\times\text{GC}$ such that (i, h)-agents have access to dwellings of type j; assume that (i, i, h) and $(i, 0, h)$ for all $(i, h)\in J\times\text{GC}$ are included in V (a consumer has access to the dwelling initially occupied by him and to a dummy one).

To validate the assumption $(i, i, h)\in V$, note that the strict access rules act only in the case of putting a dwelling that belongs to or is subsidized by the state at the disposal of citizens. If a dwelling has been already allocated to some household then a change in the household's characteristics (increase of income, change in family size, and so on) may lead to an increase in fixed costs (*e.g.*, payments for public utilities) but not to eviction from this dwelling.

Let us introduce a vector $y = (y_i \mid i\in J)$, where y_i is variable if $i\in IP$ and $y_i = 0$ if $i\notin IP$ (y_i is the number of programmatic dwellings of type i). Denote by $a_i > 0$ the amount of budgetary expenses for constructing one dwelling of type i. The problem of housing market regulation from the supply side (PSR) with variables y and $x = (x_{ijh} \mid (i, j, h)\in V)$ has the form

$$\max \textstyle\sum_{i,j,h} b_{ijh}\cdot x_{ijh} \text{ s.t.} \tag{43}$$

$$\textstyle\sum_j x_{ijh} = D_{hi}, \tag{44}$$

$$\textstyle\sum_{i,h} x_{ijh} \le C_j + y_j,\ (C_0 \text{ is large enough}) \tag{45}$$

$$\textstyle\sum_i a_i\cdot y_i \le K,\ \text{(budget restriction)} \tag{46}$$

$$x \ge 0,\ y \ge 0, \tag{47}$$

$$y \text{ is an integral vector.} \tag{48}$$

Here D_{hi} is the number of (i, h)-agents and C_j is the number of dwellings of type j in situation A_0. Each vector y satisfying conditions (46) – (48) generates some initial situation $A_{\text{reg}} = A(y)$. This situation may differ from situation A_0 only in the quantities of tenantless dwellings of types $i\in IP$ with restricted (possibly) access. The right-hand side of restriction (45) expresses the number of dwellings of type j in situation $A(y)$. Quantities $a_i\cdot y_i$ determine the appropriation of funds among dwellings types; the vector x, taking into account Assumption 9, describes the allocation. Let (x^1, y^1) be some optimum solution to PSR. By substituting values y_j^1 into the problem (43) – (45), (47), we obtain the ATC problem for situation $A(y^1)$. Surely, x^1 can be replaced by any optimum solution of this problem. Theorem 10 implies that each integral (*e.g.*, basic) optimum solution of the ATC problem for the $A(y^1)$ situation determines an allocation corresponding to some standard equilibrium for this situation.

So, by solving PSR, one can find y^1 and then analyze the equilibria that is possible in situation $A(y^1)$ using criterion (40), as described in Section 6.1.

PSR gives a reasonable approach to working-out a program of municipal housing construction: the target agents may choose dwellings either in the free market or among program dwellings (they compete for these dwellings), while other agents may be better off due to decreasing prices and/or to vacating the dwellings. The target agents' housing needs usually exceed the supply of program dwellings, and so these dwellings will be occupied by those target agents who can pay more (the richest ones among the poor).

Note that PSR does not minimize expenses. Some constraints (45) with $y_j > 0$ may be not active in the optimal solution of PSR. Surely, the "unnecessary" dwellings should be excluded from the program. Such dwellings will not appear if, after solving PSR, one shall find the "true" value of the program budget, K_0, and solve PSR once again, substituting K with K_0. One can detect value K_0 as a solution of the following problem: $\min K$ under conditions (44) – (48) and $f(x) \geq f_0$, where $f(x)$ and f_0, respectively, are the objective function (43) and the best value of this function in PSR with initial program budget K.

The poorer a consumer is, the less sensitive he is to the quality of a dwelling. Under the low solvency of target agents, the following situation is possible in the equilibrium generated by the optimal solution of PSR: some target agents remain in non-acceptable dwellings and, simultaneously, either there exist tenantless dwellings suitable for these agents or the program budget is underused. Some program for subsidizing the target agents (see Section 6.3) is necessary in this case in order to stimulate their moves into acceptable dwellings.

6.3. Market regulation from the demand side

We shall study only one of the possible approaches to housing market regulation from the demand side: subsidizing the consumers. This approach can be carried out in various forms: by paying the excess to lessors; by limiting rental rates in the government sector; by providing consumers with housing coupons, vouchers, certificates (USA) or "housing money" (Germany), see "Жилищная экономика" (1996, p. 189 – 195), "Жилищная политика ..." (1998, p. 41, 42, 100).

The designer of a program subsidizing consumers should decide whom, to what extent, and with what aim it is necessary to subsidize. The main goal of subsidizing, as a rule, is to provide socially acceptable dwellings to those who are not able to buy or rent such dwellings without a subsidy. Preventing the decline of housing facilities is the spillover result and, often, one of the program goals.

Consider the forecasted equilibrium e_0 (Section 6.1). Taking into account Lemma 3, we can suppose that the compatible equilibrium price for a dwelling, i, tenantless in this equilibrium, is $\psi_{g(i)}\, i$ if $g(i)\in G_2$ and zero if $g(i)\in G_1$. A zero price does not mean that some consumer h may receive dwelling i free of charge: he has to pay fixed expenses q_{hi}. The possible strategies of an agent owning a tenantless dwelling are: conservation, conversion, and relinquishment of the dwelling. The dwelling will be withdrawn from the market (forever, in two last cases). The worse thing is that relinquishment of a housing property may "trigger a chain reaction." Thus, a whole city district may turn into slums ("Жилищная экономика", 1996, p. 120 – 123). Though some of the dwellings tenantless in e_0 possibly fit into the sanitary standards and can provide socially acceptable conditions to low-income households within the minimum subsidy. Thus the program designer should first pay attention to consumers dislodged into unacceptable dwellings as well as to dwellings that are not in demand. Such consumers and dwellings can be detected while analyzing e_0, as well as the equilibria $e_{\max}$ and $e_{\min}$, the best and the worst according to the social criterion (Section 6.1).

Assumption 11. An (i, h)-agent can get a subsidy of type (i, j, h) (or (i, j, h)-*subsidy*) only under the condition that he chooses a dwelling of type j.

Let δ_{ijh} be the value of an (i, j, h)-subsidy, GC and GS the sets of all consumers and, respectively, of all suppliers in situation A_0. If $h\in$GC, then it is natural to assume that $\delta_{ijh} = 0$ if $j\notin D(h)$ or $j = 0$ (moves into unacceptable dwellings and outside the considered market are not subsidized). If $h\in$GS, then a $(0, h)$-agent (a supplier of group h) can choose a dwelling of type $\tau(h)$ or 0. Assume that $\delta_{0\tau(h)h} = 0$ (a supplier is not stimulated to leave the market). A subsidy of type $(0, 0, h)$ is worthwhile if it is conditioned with a decrease, by the value of the subsidy, in the reservation price of the dwelling owned by the supplier. Nevertheless, to simplify the subsequent reasoning, we put $\delta_{00h} = 0$ for $h\in$GS. It is not a restrictive assumption: instead of a subsidy to a supplier of group h, one could give the $(i, \tau(h), g)$-subsidy of equal value to some consumer g. To obtain this subsidy, the consumer should choose a dwelling of type $\tau(h)$ and, consequently, "to transfer" the subsidy to some supplier of group h. In terms of the ATC problem, a $(0, 0, h)$-subsidy with $h \in \mathrm{GS}$ decreases $\psi_{0\tau(h)h}$, and therefore increases $b_{i\tau(h)g}$ for all i and $g \in \mathrm{GC}$.

The rules of access we describe by a set, V, of triplets $(i, j, h) \in I^2 \times \mathrm{GC}$. As in Section 6.2, we assume that V includes the triplets $(i, 0, h)$ and (i, i, h) for all $(i, h) \in I \times \mathrm{GC}$.

Under the assumption made, we shall think that for each $t = (i, j, h) \in V$, Regulator specifies the value $\delta_t \geq 0$ of each t-subsidy and the number of t-subsidies.

The occupation of vacant dwellings by dislodged agents is the most natural variant of a program. More complicated variants may be also considered. Assume that $i \in D(h) \setminus D(g)$, $j \in D(h) \cap D(g)$, an (i, h)-agent, A, chooses a dwelling of type j in the equilibrium e_0 since the occupied dwelling of type i is too expensive for him, and a (k, g)-agent, B, chooses some non-acceptable dwelling. Then the (i, i, h)-subsidy may be offered to agent A (in order that he remains in his initial dwelling) and the (k, j, g)-subsidy may be offered to agent B. In other words, one may try to weaken agent A and support agent B in the competition for dwelling j. But, first, some other agents may compete for this dwelling, and second, i may be acceptable for other target agents. Therefore, a model for analyzing the consequences of any suggested policy is necessary.

Equilibrium e_0 corresponds to some optimal solutions x and (γ, π) of the AT and AT* problems. Repeating the reasoning of Lemma 4, it is easy to show that $\gamma \geq 0$. The triplet (x, γ, π) provides us with important (but unfortunately not complete) information for determining the values of δ_{ijh}. Suppose that $(i, j, h) \in V$, $x_{ikh} > 0$ (some (i, h)-agents choose dwellings of type k), and $j \neq k$. Then (see constraints of the AT* problem) $\gamma_{ih} + \pi_k = a_{ikh}$ and $\gamma_{ih} + \pi_j \geq a_{ijh}$. An (i, h)-agent g certainly will not choose j if δ_{ijh} will be less than $\gamma_{ih} + \pi_j - a_{ijh} = \gamma_{ih} + (\pi_j + q_{gj}) - b_{gj} \geq 0$, where b_{gj} is agent g's reservation price for a dwelling of type i; the bracketed expression is the full annual value of a dwelling of type j for agent g. For example, in the case of rental dwelling j, if $\gamma_{ih} = 0$ and b_{gj} is no more than 30% of w_g, then we have the known principle: "Subsidy = Actual rent (for a standard dwelling) — 0.3·Income," see "Жилищная экономика" (1996, p. 190, 191). Note that in Russia, "the own expenses for housing and public utilities within the social mark of the living space and standards of public utilities consumption for those citizens who have total per capita household revenue lower than the established cost of living, should not exceed half of the minimal rate of remuneration established by federal legislation" (the Law of Russian Federation "On foundations of the federal housing policy" in the version of the Federal laws of 12.01.96 № 9-ФЗ, of 21.04.97 № 68-ФЗ, of 10.02.99 № 29-ФЗ, of 17.06.99 № 113-ФЗ).

On the other hand, $\delta_{ijh} \geq \gamma_{ih} + \pi_j - a_{ijh} = (a_{ikh} - \pi_k) - (a_{ijh} - \pi_j)$, that is, (i, j, h)-subsidy should provide an agent, if he chooses j, utility not less than that in the case of choice k. In fact, the (i, j, h)-subsidy should make j more useful than k to (i, h)-agents, in order that an agent "feels the difference": $\delta_{ijh} = \gamma_{ih} + \pi_j - a_{ijh} + \varepsilon$, $\varepsilon > 0$. In particular, if $x_{ijh} > 0$, then, for (i, h)-agents, the dwellings of type j are available without subsidy and are equivalent to dwellings of type k. Then the (i, j, h)-subsidy is minimal $(\delta_{ijh} = \varepsilon)$; it is useful if either $k \notin D(h)$, or there are tenantless dwellings of type j, or the dwellings of type k are "needed" for other agents.

Consider the special case where the value of a subsidy may be substantiated. Suppose $\psi_{aj} \geq \psi_{bj}$ for all j, $a \in G(h, i)$, $b \in G(g, i)$ (fixed costs being equal, this is true if, for example, the agents of groups h and g have the same preferences, but the income of the agents of group h is greater than that of the agents of group g). Let γ be a vector of dual prices for constraints $\sum_j x_{ijh} = D_{hi}$ in the problem (40) – (42). If $\gamma_{ih} > \gamma_{ig}$, then the "transfer" of some (i, g)-agent into group h will increase the maximum of social criterion (40) by $\gamma_{ih} - \gamma_{ig}$. By setting $\delta_{ijg} = \psi_{aj} - \psi_{bj}$ (this difference does not depend on a and b, see Section 4) for all j, we can implement such a "transfer."

Note also that an (i, j, h)-subsidy may be expedient if the corresponding constraint (41) is not active in the AT problem and has a positive dual price in the problem (40) – (42).

Thus one can estimate the advisability of subsidizing some moves and the values of some subsidies. And then, using this information, one can establish a subsidizing program. Assume that value δ_t (possibly, zero) and the number of t-subsidies are defined for all $t \in V$. For each $(i, j, h) \in V$, let us combine all recipients of (i, j, h)-subsidies into one group. By Assumption 11, in the new market situation A_{reg}, the pure utility of a dwelling of type j for a consumer who has obtained an (i, j, h)-subsidy (corresponds to parameter a_{ijh} in situation A_0) is equal to $a_{ijh} + \delta_{ijh}$. Now one can construct the AT(A_{reg}) problem, calculate the best value of its objective function, and, using the problem analogous to (40) – (42), determine $e^{\text{reg}}_{\max}$ and $e^{\text{reg}}_{\min}$: the best and the worst equilbria, from the Regulator's point of view, for situation A_{reg}. By comparing these equilibria one can evaluate the considered program. Thus, varying the subsidies in size and number, one can fit an expedient program of housing subsidies empirically. Let us go to the description of a model that determines the number of subsidies of each type if the values δ_t for all $t \in V$ are given.

Suppose $g \in G(h, i)$, $k \in I(j)$; the direct subsidies for purchasing or renting dwellings of type j by (i, h)-agents increase b_{gk}, the indirect subsidizing through payments to owners of such dwellings decreases q_{gk} (some owner's expenses can be included in q_{gk}) and/or $\psi_{g(k)k}$. In any case, b_{ijh} in the objective function of the ATC problem increases by δ_{ijh}. Let us use the total pure utility of subsidized dwellings for recipients (including zero subsidies) as a criterion for the choice of a subsidy program. Denote by y_t (variable) the number of granted t-subsidies and put $y = (y_t \mid t \in V)$. The problem of housing market regulation from the demand side (PDR), which determines the best (under the chosen criterion) subsidizing program y, has the form

$$\max \textstyle\sum_{t \in V} (b_t + \delta_t) \cdot y_t \text{ s.t.}$$

$$\textstyle\sum_j y_{ijh} = D_{hi};\ \sum_{i,h} y_{ijh} \le C_j;\ \sum_t \delta_t \cdot y_t \le K;\ y_t \ge 0, \text{ integers.} \qquad (49)$$

Under conditions (49), the number of subsidies allocated to (i, h)-agents does not exceed the number of these agents; the number of subsidies conditional upon moving into dwellings of type j does not exceed the number of these dwellings; the budget restriction is satisfied. Put $U_0 = \{t \mid \delta_t > 0\}$. Let us assume that some program y has been accepted and the subsidized agents have been selected (but an (i, h)-agent will obtain his (i, j, h)-subsidy only if he will choose a dwelling of type j). A new market situation $A_{\text{reg}} = A(y)$ appears. In this situation, for each $t \in U_0$, y_t potential recipients of the t-subsidy form a new *t-group* of consumers; and those (i, h)-agents who didn't get a subsidy, for each $t = (i, j, h) \notin U_0$, form a *t-subgroup* of the group h; this subgroup consists of y_t potential "recipients" of the t-subsidy with $\delta_t = 0$. Notice that an (i, h)-agent being selected as a recipient of a non-zero (i, j, h)-subsidy can, nevertheless, choose a dwelling of type $k \ne j$.

We define the pure utility $a_t(k)$ of a dwelling of type k for members of the t-group or t-subgroup (*t-agents*) as follows: if $t = (i, j, h)$, then $a_t(k) = b_{ikh}$ whenever $k \ne j$, and $a_t(j) = b_t + \delta_t$ (for (i, h)-agents, the subsidy of type (i, j, h) increases the utility of dwellings of type j by δ_{ijh} and does not change the utilities of other dwellings). Put $W(i, h) = \{(i, j, h) \mid \delta_{ijh} = 0\}$.

The ATC problem for situation $A(y)$ has the form

$$\max \textstyle\sum_{t,k} a_t(k)\cdot x_t(k) \text{ s.t.} \tag{50}$$

$$\textstyle\sum_k \sum_{(i,j,h)\in W(i,h)} x_{ijh}(k) = \sum_k \sum_{(i,k,h)\in W(i,h)} y_{ikh}, \tag{51}$$

$$\textstyle\sum_k x_t(k) = y_t \text{ for } t\in U_0, \tag{52}$$

$$\textstyle\sum_t x_t(k) \le C_k,\ x_t(k) \ge 0, \tag{53}$$

where $x_t(k)$ is the number of t-agents choosing dwellings of type k.

Condition (51) corresponds to constraint (38) for a group consisting of all t-subgroups with $t\in W(i, h)$: the number of agents in this group should be equal to the number of dwellings chosen by them. Condition (52) corresponds to constraint (38) for the t-group.

The allocation that would appear if all t-agents used t-subsidies corresponding to a plan of subsidizing y we describe with vector $x(y) = (x_t(k, y) \mid t\in V, k\in I)$, where $x_{ijh}(k, y) = 0$ if $k \ne j$, and $x_{ijh}(j, y) = y_{ijh}$.

Theorem 15. Let y be an optimal solution of PDR. Then $x(y)$ is an optimal solution to problem (50) – (53). Under Assumption 11, $x(y)$ determines some equilibrium allocation for situation $A(y)$.

So, PDR determines both a rational subsidizing program and one of those equilibria that may appear in the case of implementing this program.

6.4. Concluding comments

Voluntariness of choice. The arguments in Sections 6.2 and 6.3 show that some standard equilibrium e_{reg} in the considered market corresponds to any optimal solution of PSR or PDR. Suppose that some equilibrium e arose as a result of implementing a housing program. The voluntariness of market agents' choices is restricted only by access rules, and the numerosity of equilibria does not allow us, generally speaking, to assert that $e = e_{\text{reg}}$. Nevertheless, in equilibrium e, the non-target agents do not occupy programmatic dwellings (because of access restrictions), and each (i, j, h)-subsidy is granted to an agent only if he has chosen an acceptable dwelling of type j. Theorem 2 implies that an arbitrary system of equilibrium prices equilibrates any equilibrium allocation. Therefore, the dwelling a target agent would occupy in e_{reg} is the best choice (possibly, not a unique best) for him in e too. It seems obvious that a close approximation to one of the equilibria e_{reg} may be achieved by combining the construction of programmatic dwellings, housing subsidies, and advance advertizing.

Variants of the Regulator's objective function. We have considered (Section 6.1) a linear in variables of the AT problem Regulator's objective function. In an analogous way, one can analyze an objective function, R, linear in variables (γ and π) of the AT* problem (*e.g.*, the sum of prices for dwellings of some types). If Q is the polytope of the AT* problem, then $Q_0 = \{(\gamma, \pi) \in Q \mid \sum_{h,n} D_{hn} \cdot \gamma_{hn} + \sum_k C_k \cdot \pi_k \le f_0\}$ is the optimal face of this polytope. Thus, Regulator can compare the best and the worst values of R over all equilibrium price systems by solving the corresponding linear programming problems over Q.

Coordination of housing programs. Market regulation from the supply side through municipal housing construction without subsidizing the consumers is reasonable if there exists a solvent demand of target consumers for programmatic dwellings at prices recouping expenses. Disregarding this condition results in the "construction of extra quantities of new flats offered for sale by municipal authorities in some regions, while … some groups of the population are strongly in need of dwellings" ("Жилищная политика …", 1998, p. 210). On the other hand, subsidizing the consumers, not supported by a housing construction program, is meaningful if there are enough tenantless or vacant (withdrawn) dwellings appearing on the market. Otherwise, either subsidies will not be used or the prices of the dwellings acceptable for target consumers will increase. Consequently, a joint program for market regulation is necessary. Maybe, combining PSR and PDR, it is possible to construct a model for selecting such program.

Government sector. Considering the budget financing of housing programs we, in fact, introduce the sector of government dwellings with limited access and regulated prices into the initial market model. The price regulation in this sector is a form of subsidizing. The local authorities can establish different prices for different groups of consumers.

Financing the housing programs. Under the above considered approaches to regulating the housing market, the regulation costs (parameter K in problems PSR and PDR) are covered by the budget to a great extent. In developed countries, as a rule, the central budget subsidizes local budgets to implement local housing programs, see Бессонова (1993, p. 57, 139). The Russian federal program "Dwelling" (approved by Council of Ministers and Government of the Russian Federation Act No 595 of June 20, 1993) indicates sources of means for subsidizing and creating housing facilities for social use. The federal sources are the following: the budget, state guaranteed credits, funds of the State Employment Fund (establishment of a public works program in the housing sphere), a share of federal export quotes, *etc*. The local sources include the following: the tax on profits from selling state dwellings, a share of the real estate tax, a share of the profit from selling prepared building plots by local authorities.

After further revision of this program (Basic directions of the new stage in the state goal-oriented program "Dwelling," approved by the President of the Russian Federation, Decree No 431 of 29.03.96), the central budget share in financing local housing programs has been strongly reduced. In particular, enterprises and organizations managing municipal housing facilities should grant housing subsidies to citizens at the cost of their own means. As additional sources of financing, the means from selling some constructed dwellings at commercial auctions and the profit from municipal rents have been indicated. A part of profits obtained from increasing prices for housing and public utilities can be used for compensating payments for housing and public utilities (Standing on procedure of granting compensation (subsidies) for housing and public utilities payments; approved by the Government of the Russian Federation Act № 595 of 18.06.96). Therefore, the local authorities can finance "loss-making" housing programs at the expense of *profitable* programs or activities.

When selling the dwellings from municipal housing facilities in a free market, the local authorities play the role of a profit-maximizing supplier. The AT problem allows one to select a seller's maximal reservation price such that his own dwelling would be in demand at this price. If municipal authorities finance the construction of some dwellings in order to sell them in a free market, then PSR with unrestricted access ($V = I^2 \times H$) may help allocate the means among dwellings types.

In many cities, some part of the dwellings constructed by private investors comes to be at the disposal of local authorities free of charge. Such a rule has been enacted, for instance, in St. Petersburg ("Жилье", 1998, p. 53). Also, in Novosibirsk, the city authorities receive free of charge 5% of the new dwellings (Смородинов, 1999). The expected private housing construction volume determines the total space, S, of dwellings to be transferred to the local authorities. This total space can be distributed among types of dwellings with the help of the PSR problem, measuring the regulation expenses not in money but in dwelling-space (*i.e.*, by letting a_i be the space of one dwelling of type i and $K = S$).

Developing the approach suggested by Gustafsson, *et al*. (1980, p. 88, 89), the duals can be used for "negative subsidizing" some bargains in the housing market. Let x and (γ, π) be some optimal solutions of the AT and AT* problems determining some equilibrium e_0 for situation A_0. Assume that in this equilibrium some (i, h)-agents choose dwellings of type j and have in this connection a positive consumer's surplus: $\gamma_{ih} = a_{ijh} - \pi_j > 0$.

Let us select δ such that $0 < \delta < \gamma_{ih}$ and put $J(i, h) = \{k \in I \mid \gamma_{ih} = a_{ikh} - \pi_k\}$. $J(i, h)$ is the set of all dwellings types equivalent to j for (i, h)-agents at prices $p = r(\pi)$ (see Section 1.1). Put $\beta = (\beta_{kg} \mid (k, g) \in I \times H)$, where $\beta_{ih} = \gamma_{ih} - \delta$ and $\beta_{kg} = \gamma_{kg}$ if $(k, g) \neq (i, h)$. It is easy to see that x and (β, p) are optimal solutions of the AT and AT* problems for situation B, obtained from A_0 by replacing a_{ikh} with $a_{ikh} - \delta$ for $k \in J(i, h)$. In other words, if you decrease a_{ikh} by δ for all $k \in J(i, h)$, then (i, h)-agents will not change their choices and both the allocation and price vector will be the same in the new equilibrium as in e_0. In this case, from the bargain corresponding to a triplet (i, k, h), $k \in J(i, h)$, the local authorities may have a budgetary gain of annual income δ for financing housing programs. For this purpose, it is enough to introduce, with respect to (i, h)-agents, some additional fixed payment for dwellings of types $k \in J(i, h)$. It may be either the recurrent payment δ or the non-recurrent payment $\delta \cdot \rho^{-1}$.

Thus, the models offered above can be useful for working-out profitable programs and activities in housing markets. To simplify the analysis one can assume that profitable and lossmaking programs are separated in time: the profits obtained by a local budget within the current period of time is invested in lossmaking programs of the next period, the local authorities first accumulate means and then spend them.

Approximate solutions of PSR and PDR. Formally, PSR and PDR are integer linear programming problems of a transportation type with one knapsack restriction. They are NP-hard: the problem "INTEGRAL KNAPSACK" (Papadimitriou and Steiglitz, 1982, Section 15.7) is reducible to each of them. Thus, solving these problems under actual dimensions is very cumbersome. The author has suggested a method for approximating the solution of PSR. The ponderous description and substantiation of the method are not included in this publication and will be published in the article by Khutoretsky (forthcoming).

The idea is to analyze non-integer components of an arbitrary *bfs* (x, y) for the linear relaxation of PSR (denote this linear relaxation by LPSR). It turns out that these components create some specific structure, a *route*. A route consists of all links (i, j, h) such that x_{ijh} is not integer. A link (i, j, h) has the *beginning* i, the *source* (i, h), and the *end* j. If two links of a route are *neighboring* then either they have the common source (or the beginning, or the end), or the beginning of one of these links coincides with the end of the other. Therefore, the *direct* and *inverse* links can be distinguished in a route. The beginnings and the ends of links are *vertices* of a route.

The route μ created by all non-integer components of x is not closed. Only the components corresponding to extreme links of μ can be non-integer in the vector y.

A feasible solution (x, y) of the LPSR problem is called *economical* if, for any i, either $y_i = 0$ or the corresponding constraint (45) is active (unused dwellings are not financed). Given a basic optimal solution to LPSR, one can construct an economical basic optimal solution of this problem.

Taking account of the results listed above, let us consider that (x, y) is some optimal, basic, economical, non-integral solution of LPSR, and μ is the route created by this solution.

It appears that in vector x, the components corresponding to the links of the same (direct or reverse) orientation have the same fractional part, and the sum of the fractional parts of two components corresponding to oppositely orientated links is equal to unity. Besides, the fractional parts of the non-integer components in vectors x and y are adjusted such that it is possible to "round-off" the solution (x, y) correctly along route μ. As a result, we shall obtain the optimal solution of LPSR with the right-hand side of constraint (46) replaced by some number K_1. Moreover, if i and j are the extreme vertices of the route μ, then $|K_1 - K| \le 0.5 \cdot \max\{a_i, a_j\}$. That is, K_1 differs from K by no more than half the cost of some financed dwelling. Obviously, if an optimal solution of LPSR is integral, then it is an optimal solution to PSR with the same right-hand side of constraint (46). In other words, we can solve PSR with a slightly modified program budget. It seems that such accuracy is enough for economic applications (the limitation K is not rigid, as a rule). We can find an optimal distribution (among types of dwellings) of the sum, which differs from the limit of means assigned for housing construction by not more than half the cost of the most expensive financed dwelling.

Applying the described approach to PDR, we consider its linear relaxation LPDR. Put $\varepsilon_{ijh} = \delta_{ijh} - \delta_{iih}$ for $i \ne j$. As we have done for PSR, each non-integral basic optimal solution y of PDR can be "correctly rounded-off" as to obtain an integral optimal solution y^1 to LPDR with the slightly modified right-hand sides of constraints. It is enough to replace the right-hand side of the condition $\sum_t \delta_t \cdot y_t \le K$ with some number K_1 such that $|K_1 - K| \le \max\{\varepsilon_t \mid y_t > 0\}$, and to increase by unity the right-hand side of one condition $\sum_{i,h} y_{ijh} \le C_j$ (say, C_k). It is clear that y^1 is an optimal solution to PDR with the same (modified) right-hand sides of constraints.

This vector describes a rational distribution (in the form of subsidies) of a sum, differing from K by not more than the value of one subsidy. When knowing y^1, one can construct the corresponding equilibrium allocation (Theorem 19) for a hypothetical situation A_1 with one additional (in comparison with the forecasted situation A_0) tenantless dwelling of type k. Some standard equilibrium with this allocation would be possible in situation A_1 under the subsidizing policy y^1.

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Theorem 1. If $g \quad G_1$, then $b_{gi} = e_{gi} \quad e_{g0} + q_{g0}$ for all i.

Proof. Condition (4) is equivalent to $b_{gi}(p) \quad e_{gi} \quad e_{gd(g)}$, (5) is equivalent to $b_{gi}(p) \quad e_{gi} \quad e_{gd(g)}$. Therefore, $Db_{gi} = \min\{e_{gi} \quad e_{gd(g)}, w_g \quad y_g^0 \quad q_{gd(g)}\}$. It follows from (2) that $Db_{gi} = e_{gi} \quad e_{gd(g)}$. Then $b_{gi} = e_{gi} \quad e_{g0} + q_{g0}$ is a unique solution of system (6). *Q.E.D.*

Lemma 1. Condition (9) is equivalent to $y_{gj(z,g)} \quad y_{gi} \quad c(p_{j(z,g)}) \quad c(p_i)$ for all g and $i \quad J_g$.

Proof. Put $j = j(z, g)$. Condition (9) is equivalent to $v_{gj}(p) \quad v_{gi}(p)$ for all g and $i \quad J_g$. Now the statement of the lemma follows from (1), Theorem 1, and the definition of y_{gi}. *Q.E.D.*

Lemma 2. Let (z, p) be some equilibrium, and put $p = c(p)$.

(a) For all g, $u_g(z) = w_g + y_{gj(z,g)} + p_{d(g)} \quad p_{j(z,g)}$; if $g \quad G_2$, then $u_g(z) = w_g + \max\{y_{gd(g)}, p_{d(g)}\}$.

(b) If $g \quad G_1$, then $y(z, g) \quad y_g^0$.

(c) If $g \quad G_2$, then $p_{d(g)} \quad y_{gd(g)}$ whenever $d(g) \quad \{j(z, h) \mid h \quad G_1\}$ and $y_{gd(g)} \quad p_{d(g)}$ otherwise.

Proof. Let (z, p) be an equilibrium, and put $p = c(p)$. Statement (a) is obvious. Suppose $g \quad G_1$. It is easy to see that $y_{gi}(p)$ is a monotone decreasing function of $b_{gi}(p)$. Thus $y_{gi}(p) \quad y_g^0$ if $b_{gi}(p) \quad Db_{gi}$. Then $y_{gi}(p) < y_g^0$ implies $(c(p_i) + q_{gi}) \quad (c(p_{d(g)}) + q_{gd(g)}) > Db_{gi} = b_{gi} \quad b_{gd(g)}$, whence $y_{gi} \quad y_{gd(g)} < c(p_i) \quad c(p_{d(g)})$. If $i = j(z, g)$, then the last inequality conflicts with (9) (Lemma 1), and (b) is proved. To prove (c) let us take some $g \quad G_2$. If $d(g) \quad \{j(z, h) \mid h \quad G_1\}$,

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[1] Here the complete proofs of all results included in the printed version of the publication are given. The numbering of formulas is continued.

then $j(z, g) = 0$ by (8). From Lemma 1 with $i = d(g)$, taking into account that $p_0 = y_{g0} = 0$, we obtain $0 \quad y_{gd(g)} \quad p_{d(g)}$. If $d(g) \quad \{j(z, h) \mid h \quad G_1\}$, then, by (10), either $j(z, g) = d(g)$ or $p_{d(g)} = 0$. If $j(z, g) = d(g)$, then $y_{gd(g)} \quad p_{d(g)}$ follows from Lemma 1 with $i = 0$; otherwise $y_{gd(g)} \quad p_{d(g)}$ since $y_{gd(g)} \quad 0$ for $g \quad G_2$ by the definition. *Q.E.D.*

Lemma 3. Assume that $p \quad P$, $p = c(p)$, and the vectors p^1, p^1 are defined as follows: if $g(i) \quad G_2$, then $p_i^1 = \max\{p_i, y_{g(i)i}\}$, p_i otherwise; $p_i^1 = r(p_i^1)$ for all $i \quad I$. If (z, p) is an equilibrium, then (z, p^1) is an equilibrium too.

Proof. Suppose $p \quad P$ and $j = j(z, g)$. Consider $g \quad G_1$. Either $g(j) \quad G_2$ and $p_j \quad y_{g(j)j}$ by Lemma 2, or $g(j) \quad G_1$, or $j = 0$; in any case $p_j^1 = p_j$. Applying Lemma 1 to equilibrium (z, p), we obtain $0 \quad (y_{gj} \quad y_{gi}) \quad (p_j \quad p_i) \quad (y_{gj} \quad y_{gi}) \quad (p_j^1 \quad p_i^1)$ for all i. Consider $g \quad G_2$. If $p_{d(g)}^1 = p_{d(g)}$, then $p_i^1 = p_i$ for all $i \quad J_g$, in particular, $p_j^1 = p_j$. Then, as in the previous case, $y_{gj} \quad y_{gi} \quad p_j^1 \quad p_i^1$ for $i \quad J_g$. Assume that $p_{d(g)}^1 \quad p_{d(g)}$. Then $p_{d(g)} < y_{gd(g)}$, $j = d(g)$ by (9), $p_{d(g)}^1 = y_{gd(g)}$ and $y_{gj} \quad y_{g0} = p_j^1 \quad p_0^1$. So, $y_{gj} \quad y_{gi} \quad p_j^1 \quad p_i^1$ for all g and $i \quad J_g$, thus condition (9) is true for (z, p^1) by Lemma 1. If $p_i^1 \quad p_i$, then $g(i) \quad G_2$ and, as were shown above, $j(z, g(i)) = i$. Consequently, $p_i^1 = p_i$ for $i \quad \{j(z, g) \mid g \quad G\}$, and (z, p^1) satisfies condition (10). *Q.E.D.*

Lemma 4. If (a, p) is the optimal solution of problem T*, then $a \quad 0$ and $p \quad P$.

Proof. Constraint (14) is not active for any optimal solution of T by the choice of S_0, thus $p_0 = 0$ and $p \quad P$; (6) implies $y_{g0} = 0$. Then, in (17) with $i = 0$, we obtain $a_g \quad 0$. *Q.E.D.*

Theorem 2. If (z, p) is an equilibrium, then $x(z)$ is a basic optimal solution of T and $(a(z,p), c(p))$ is an optimal solution of T*. If x is a basic optimal solution of T and (a, p) is an optimal solution of T*, then $(z(x,p), r(p))$ is an equilibrium.

Proof. (1) Let (z, p) be an equilibrium. Put $x = x(z)$. It follows from (8) that x belongs to the polytope defined by constraints (12) (15); this polytope is imbedded into unit cube and x is an integral vector, thus x is a vertex of the polytope and x is a *bfs* to problem T. Take $g \ \ G$, $i \ \ J_g$. Put $z^1 = (y_{gi}(p), i) \ \ Z_g$. It follows from (9) and (1) that $a_g(z, p) = u_g(z) \ \ w_g \ \ c(p_{d(g)}) \ \ u_g(z^1) \ \ w_g \ \ c(p_{d(g)}) = y_{gi} \ \ c(p_i)$. Thus $(a(z,p), c(p))$ is a feasible solution of T*.

We claim that the functions (16) and (11) of $(a, p) = (a(z,p), c(p))$ and $x = x(z)$, respectively, take the equal values. Indeed, put $A = \{j(z, g) \mid g \ \ G\}$. Note that, by (10), $p_i = 0$ if $i \ \ A$. Using Lemma 2, we obtain $\ _{g,i}\, y_{gi} \times x_{gi} = \ _g\, y_{gj(z,g)} = \ _g\, [u_g(z) \ \ w_g \ \ c(p_{d(g)}) + c(p_{j(z,g)})] = \ _g\, a_g(z, p) + \ _{i \ A}\, c(p_i) = \ _g\, a_g(z, p) + \ _i\, c(p_i)$. Thus $x(z)$ and $(a(z,p), c(p))$ are optimal solutions of T and T* respectively (by the first duality theorem).

(2) Suppose x is a basic optimal solution of problem T, (a, p) an optimal solution of T*. It is pointed out in section 2.1 that x is an integral vector. Put $z = z(x, p)$ and $p = r(p)$; surely, $(z, p) \ \ Z \ \ P$. Take $g \ \ G$. By the definition of $z(x, p)$ we have $j(z, g) = k(x, g)$. By the second duality theorem, $a_g + p_{j(z,g)} = y_{gj(z,g)}$ (since $x_{gk(x,g)} = 1$), and, taking into account (17), $y_{gj(z,g)} \ \ p_{j(z,g)} \ \ y_{gi} \ \ p_i$ for all $i \ \ J_g$. The last inequality is equivalent to (9) (Lemma 1). Condition (8) is true for z by the definition of $z(x, p)$. If $i \ \ \{j(z, g) \mid g \ \ G\}$, then (13) is not active for i, $p_i = p_i = 0$, and (10) is true for $z(x, p)$. Then (7) is true as well (Mas-Collel, Whinston, Green, 1995, Lemma 10.B.1, p. 316). *Q.E.D.*

Corollary 1. If $z \ \ FD$, then $\ _{g \ G}\, u_g(z) = \ _g\, w_g + \ _{g,i}\, y_{gi} \times x_{gi}(z)$; in particular, if $z \ \ E$, then $\ _{g \ G}\, u_g(z) = F_0$.

Proof. Suppose $z \ \ FD$. Using (7), we obtain: $\ _g\, u_g(z) = \ _g\, [y(z, g) + b_{gj(z,g)}] = \ _g\, [b_{gj(z,g)} \ \ q_{gj(z,g)} + w_g] = \ _g\, [y_{gj(z,g)} + w_g] = \ _g\, w_g + \ _{g,i}\, y_{gi} \times x_{gi}(z)$. If $z \ \ E$, then, by Theorem 2, $\ _{g,i}\, y_{gi} \times x_{gi}(z) = f_0$ and, consequently, $\ _g\, u_g(z) = F_0$. *Q.E.D.*

Corollary 2. If $z \ \ FD$, $x(z)$ is a basic optimal solution of problem T, and there exists an optimal solution of problem T* such that $u_g(z) = a_g + w_g + p_{d(g)}$ for all $g \ \ G$, then $(z, r(p))$ is an equilibrium.

Proof. Suppose z and (a, p) satisfy the conditions of Corollary 2, $p = r(p)$, and $z^1 = z(x(z), p)$. By Theorem 2, (z^1, p) is an equilibrium, and obviously $z(z^1) = z(z)$. If $i = j(z, g) = j(z^1, g)$, then $x_{gi}(z) = 1$ and the corresponding constraint (17) is active; thus $a_g = y_{gi} \quad p_i$. The definition of $z(x, p)$ and Lemma 2 imply $y(z^1, g) = w_g + p_{d(g)} \quad p_i \quad q_{gi} = w_g + p_{d(g)} + a_g \quad (y_{gi} + q_{gi}) = u_g(z) \quad b_{gi} = y(z, g)$. Then $z = z^1$, hence (z, p) is an equilibrium. *Q.E.D.*

Lemma 5. If $z \quad \tilde{N}$, then $_{g \quad G} u_g(z) = F_0$ and $u_g(z) \quad w_g + y_{gd(g)}$ for all $g \quad G$.

Proof. Suppose $z \quad \tilde{N}$ and z^1 is some equilibrium distribution. Then $_g u_g(z)$ $_g u_g(z^1) = F_0$ by Corollary 1. Assume that $_g u_g(z) < F_0$. Put $e = (|G| \quad ^1)\times[F_0$ $_g u_g(z)] > 0$ and $y_g = e + u_g(z) \quad u_g(z^1)$. Let us consider distribution $z^0 = (z_g^0 \mid$ $g \quad G)$, where $z_g^0 = (y_g + y(z^1, g), j(z^1, g))$. Obviously, $_g y_g = 0$ and z^0 FD, while $u_g(z^0) = u_g(z^1) + y_g = u_g(z) + e$. So, coalition G blocks z contrary to the assumption $z \quad \tilde{N}$. Therefore $_g u_g(z) = F_0$. If $u_g(z) < w_g + y_{gd(g)}$, then coalition $Q = \{g\}$ blocks z with distribution $(w_g \quad q_{gd(g)}, d(g))$ FD(Q); therefore $u_g(z)$ $w_g + y_{gd(g)}$ for all $g \quad G$. *Q.E.D.*

Lemma 6. Assume that $z \quad \tilde{N}$, $x = (x_{gi} \mid g \quad G, i \quad J_g)$, and $c_{gi} = y_{gi} \quad u_g(z) + w_g$ for all $g \quad G$, $i \quad J_g$. Then the following problem is bounded:

$$\max \quad _{g,i} x_{gi} \times c_{gi} \quad \text{s.t.} \tag{18}$$

$$x \quad 0 \quad \text{and} \quad _g x_{gi} \quad _j x_{g(i)j} \quad 0 \quad \text{for } i \quad 0. \tag{19}$$

Proof. Suppose $z \quad \tilde{N}$ and problem (18) (19) is not bounded. Then there exists a feasible solution x of this problem such that $_{g,i} x_{gi} \times c_{gi} > 0$. Take such x; it is clear that $\{(g, i) \mid x_{gi} > 0\}$. A sequence $l = h(1), i(1), \quad , h(n), i(n)$ will be called *maximal* with respect to x if the following conditions hold: $h(s)$ G, $i(s) \quad J_{h(s)}$, and $x_{h(s)i(s)} > 0$ for all s; $h(s+1) = g(i(s))$ (and thus $i(s) \quad 0$) whenever $1 \quad s < n$; if $i(n) \quad 0$, then $h(1) = g(i(n))$; if $i(n) = 0$, then either $_g x_{gd(h(1))} = 0$ or $d(h(1)) = 0$. It follows from (19) that there exists some maximal with respect to x sequence l. Put $A(l) = \{(h(s), i(s)) \mid 1 \quad s \quad n\}$, $L(l) = \; _{(g,i) \; A(l)} c_{gi}$ (*weight* of the sequence), $e = \min\{x_{gi} \mid (g, i) \quad A(l)\} > 0$. Assume that $L(l)$ 0. In this case let us put $y_{gi} = x_{gi} \quad e$ if $(g, i) \quad A(l)$, x_{gi} otherwise. It is clear that the vector $y = (y_{gi})$ satisfies (19) and $\sum_{g,i} y_{gi} c_{gi} = \sum_{g,i} x_{gi} c_{gi} - L(l) e > 0$. But l is not a maximal sequence with respect to y since $y_{gi} = 0$ for some pair $(g, i) \in A(l)$.

Proceeding repeatedly, we shall construct a vector x satisfying (19) such that $\sum_{g,i} x_{gi} c_{gi} > 0$ and each sequence maximal with respect to x is of positive weight. Let $l = \langle h(1), i(1), \ldots, h(n), i(n) \rangle$ be such sequence. Put $e = L(l) n^{-1} > 0$, $Q = \{h(s) \mid 1 \le s \le n\}$. Then we put $j(h(s)) = i(s)$ for all s (moving of the members of coalition Q in concordance with l), $y(g) = u_g(z) - b_{gj(g)} + e$, $z_g = (y(g), j(g)) \in Z_g$ for each $g \in Q$, and $z^1 = (z_g \mid g \in Q)$. Using the definitions of e, $L(l)$, c_{gi}, and y_{gi}, we obtain $\sum_{g \in Q} y(g) = \sum_{g \in Q} [w_g - q_{gj(g)}]$. Hence $z^1 \in FD(Q)$. But $u_g(z^1) = y(g) + b_{gj(g)} = u_g(z) + e$ for all $g \in Q$, thus coalition Q blocks z, a contradiction. *Q.E.D.*

Theorem 3. $E = \tilde{N} \cap PM$. If $|G| \ge 2$, then $E \neq PM$.

Proof [2]. The inclusion $E \subseteq C$ is well-known (e.g. see Mas-Collel, Whinston, Green, 1995, p. 654). Let us prove that $C \subseteq E$. Suppose $z \in C$. Then $x(z)$ is a feasible solution to T. It follows from Lemma 5 and Corollary 1 of Theorem 2 that $\sum_{g,i} y_{gi} x_{gi}(z) = \sum_g u_g(z) - \sum_g w_g = F_0 - \sum_g w_g = f_0$, thus $x(z)$ is an integral (and, therefore, a basic) optimal solution of problem T. Put $p_0 = 0$ and consider the following system of inequalities:

$$p_0 = 0, \quad p_i - p_{d(g)} \ge y_{gi} - u_g(z) + w_g \quad \text{for } g \in G,\ i \in J_g, \qquad (54)$$

where p_i with $i \neq 0$ are variables (we place p_0 in the left sides of some inequalities for uniformity).

The problem dual to system (54) has the form (18), (19). This is bounded by Lemma 6, thus system (54) is solvable; let $(p_i \mid i \in I \setminus \{0\})$ be some solution to this system. Vector $p = (p_i \mid i \in I)$ is completely defined now. Put $a = (a_g \mid g \in G)$, where $a_g = u_g(z) - w_g - p_{d(g)}$. By (54), (a, p) is a feasible solution of problem T*. Using the definition of a_g, we obtain $\sum_g a_g + \sum_i p_i = \sum_g [u_g(z) - w_g] = f_0$ (the last equality is proved above). Then (a, p) is an optimal solution of

[2] An idea by Quinzii (1984, Theorem 3) is used in this proof.

problem T*, and $z\ \ E$ by Corollary 2 of Theorem 2.

Let us prove that $C\quad$ PM. Suppose the converse: $z\ \ C$ \ PM. From $z\ \ $PM it follows that $\ _g u_g(z^1) >\ _g u_g(z)$ for some $z^1\ $ FD. Applying Lemma 5 and Corollary 1 of Theorem 2, we obtain the impossible inequality $\ _{g,i}\, y_{gi}$⋇$_{gi}(z^1) =\ _g [u_g(z^1)\quad w_g] >\ _g [u_g(z)\quad w_g] = f_0$.

Suppose {1, 2} $\quad G$ (there are at least two agents in the market). Let us construct a distribution in PM \ E. Take $z\ \ E$. Put $y^1(1) = y(z, 1)\quad e$, $y^1(2) = y(z, 2) + e$, $y^1(g) = y(z, g)$ for $g\ \ G$ \ {1, 2}. Let $z(e)$ be the vector with components $(y^1(g), j(z, g))$ for $g\ \ G$. Obviously, $z(e)\ $ FD. Assume that there exists $z^0\ $ FD such that $u_g(z^0)\quad u_g(z(e))$ for all $g\ \ G$ with at least one strict inequality. Put $y^2(1) = y(z^0, 1) + e$, $y^2(2) = y(z^0, 2)\quad e$, $y^2(g) = y(z^0, g)$ for $g\ \ G$ \ {1, 2}. Consider a distribution, z^2, such that $z_g^2 = (y^2(g), j(z^0, g))$ for all $g\ \ G$. It is clear that $z^2\ $ FD and $u_g(z^2)\quad u_g(z)$ for all g with at least one strict inequality; this contradicts the inclusion $E\quad$ PM proved above. Hence $z(e)\ $ PM for any $e\ \ R$. If $e > u_1(z)\quad (w_1 + y_{1d(1)})$, then $u_1(z(e)) = u_1(z)\quad e < w_1 + y_{1d(1)}$, and $z(e)$ $C = E$ by Lemma 4. So, $z(e)\ $ PM \ E if $e > u_1(z)\quad (w_1 + y_{1d(1)})$. *Q.E.D.*

Lemma 7. If $r\ \ \hat{A}$, then $s(r)$ is an orderly signal and conditions (21) $\quad$ (23) are satisfied for any collection of trade offers $Q = (Q_g \mid g\ \ G)$.

Proof. **Suppose the demand constraint for dwelling i is binding to agent g G_1: $w_g(i) > w_g(t(g))$. By the definition, $t(g) = t(g, D^k)$ for some $k\quad$ 1. Then i D^k. By construction of the sets D^n for $n\quad k$, it is clear that $i\ \ \{t(h) \mid h\ \ G\}$. Thus $s\ (r, g(i)) =\quad 1$ and the constraint of supply is not binding to agent $g(i)$, therefore $s(r)$ is an orderly signal. It follows from $I^+(r, g) = \{0, d(g), t(g)\}$ that $Q_g\quad I^+(r, g)\quad \{t(g)\}$; the last relation is enough to meet conditions (21) (23).** *Q.E.D.*

Theorem 4. If $r\ \ \hat{A}$, then $z(r)$ is an equilibrium with respect to scheme r.

Proof. **Obviously, $z(r)\ $ DA($s(r)$). Let us put $Q_g(r) = (\{i\ \ J_g \mid w_g(t(g)) < w_g(i)\}$ $\{t(g)\})$ \ $\{0, d(g)\}$ and $Q(r) = (Q_g(r) \mid g\ \ G)$. It is easily seen that the conditions (25) $\quad$ (27) with $Q = Q(r)$ and (24) with $g\ \ G_2$ are true for distribution**

$z(r)$. Suppose $g \ \ G_1$. Then, for some k, $g \ \ A^k$ and $t(g) = t(g, D^k)$. The initial conditions of step k imply $D^k \ \ I^+(r, g)$. Then $w_g(t(g)) = \max\{w_g(i) \mid i \ \ D^k\}$ $\max\{w_g(i) \mid i \ \ I^+(r, g)\} = U_g(s(r))$, and $\max\{w_g(i) \mid i \ \ I^+(r, g)\} \ \ w_g(t(g))$ since $t(g) \ \ I^+(r, g)$. Therefore, (24) is true for $g \ \ G_1$. Now we need only to verify that for each $g \ \ G$, $Q_g(r)$ is agent g's effective demand with respect to signal $s(r)$. This is true for $g \ \ G_2$ because $J_g = \{d(g), 0\}$. Take $g \ \ G_1$. If $i \ \ I^+(r, g) \setminus \{d(g), 0\}$, then $i = t(g) \ \ Q_g(r)$. If $i \ \ J_g \setminus I^+(r, g)$ and $w_g(i) > w_g(j)$ for all j $I^+(r, g)$, then $i \ \ \{d(g), 0\}$, $w_g(i) > w_g(t(g))$, and thus $i \ \ Q_g(r)$. So, $Q(r)$ is a collection of effective demands with respect to signal $s(r)$. *Q.E.D.*

Theorem 5. If $r \ \ \hat{A}$ and condition (28) is true, then $z(r)$ is a unique equilibrium with respect to scheme r.

Proof. Let z be an equilibrium with respect to some scheme $r \ \ \hat{A}$. Then $z \ \ \mathrm{DA}(s(r))$. Recall that $z_g(r) = t(g)$ and r relates the signal $s(r)$ independent of Q to any collection of trade offers Q.

Consider $g \ \ G_2$; then $d(g) \ \ 0$. If $t(g) = d(g)$, then $s(r, g) = \ 1$ by the definition, and $t(g) \ \ I^+(r, h)$ for $h \ \ G_1$, thus $z_h \ \ t(g)$ for $h \ \ G_1$; now $z_g = 0$ contradicts condition (27) for z, therefore $z_g = d(g)$. If $t(g) = 0$, then (28) implies $w_g(0) > w_g(d(g))$, and $z_g = 0$ by (24). So, $z_g = t(g)$ for $g \ \ G_2$.

Take $g \ \ G_1$. The following cases are possible: $t(g) \ \ 0$, $0 = t(g) \ \ d(g)$, and $0 = t(g) = d(g)$. If $t(g) \ \ 0$, then $t(g) = d(f)$ for some $f \ \ G$, and $s(r, f) = \ 1$ by the definition. If $h \ \ G_1 \setminus \{g, f\}$, then $t(g) \ \ I^+(r, h)$, thus $z_h \ \ t(g)$ for $h \ \ G_1 \setminus \{g, f\}$. It follows from $t(g) = d(f)$ that $t(f) \ \ d(f)$, and thus (28) implies $w_f(t(f)) > w_f(d(f))$; therefore $t(g) = d(f) \ \ z_f$ by (24). So, $z_h \ \ t(g)$ for $h \ \ G_1 \setminus \{g\}$. We have proved above that $z_h = t(h)$ for $h \ \ G_2$. So, $z_h \ \ t(g)$ for $h \ \ G \setminus \{g\}$. Then $z_g \ \ t(g)$ would imply $t(g) = d(g_1) \ \ \{z_h \mid h \ \ G\} \ \ \{0\}$ contrary to condition (27) for z. If $t(g) = 0 \ \ d(g)$, then (28) implies $w_g(0) > w_g(d(g))$, and $z_g = 0$ by (24). At last, if $0 = t(g) = d(g)$, then $I^+(r, g) = \{0\}$, thus $z_g = 0$. So, $z_g = t(g)$ for all g. *Q.E.D.*

Theorem 6. FPC = $E(\hat{A})$.

Proof [3]. **1. Let z = $z(r)$, r $\hat{A}$. Then $w_g(z_g)$ = max$\{w_g(i) \mid i$ $J_g\}$ for g G_2 by (24). Assume that z FPC, $z \prec x$, and l = $g(1)$,   , $g(n)$ is the corresponding augmentative sequence. By the definition, put st(s) = k if the r-algorithm determines $t(g(s))$ = $z_{g(s)}$ = $t(g(s), D^k)$ at step k. Consider s < n. If $z_{g(s)}$ = $x_{g(s)}$, then $x_{g(s)}$ $z_{g(s+1)}$ (as z DA) and (29) implies $z_{g(s)}$ = $x_{g(s)}$ = $d(g(s+1))$, i.e. agent $g(s+1)$ joins a feasible sequence not later than at step st(s): st(s) st(s+1). If $z_{g(s)}$ $x_{g(s)}$, then $w_{g(s)}(x_{g(s)}) > w_{g(s)}(z_{g(s)})$ by (31) and $x_{g(s)}$ $D^{\mathrm{st}(s)}$ by the definition of $t(g(s))$. Furthermore, by (29), either $x_{g(s)}$ = $d(g(s+1))$ or $x_{g(s)}$ = $z_{g(s+1)}$. In the first case, the initial conditions of step st(s) imply $g(s+1)$ $A^{\mathrm{st}(s)}$; in the second case, $t(g(s+1))$ = $z_{g(s+1)}$ was defined before step st(s); in any case st(s) > st(s+1). If $x_{g(n)}$ $\{d(g(1)), z_{g(1)}\}$, then the analogous reasoning gives st(n)   st(1) with strict inequality whenever $w_{g(n)}(x_{g(n)}) > w_{g(n)}(z_{g(n)})$. So, if $x_{g(n)}$ $\{d(g(1)), z_{g(1)}\}$, then st(1) ... st(n) st(1) with at least one strict inequality by (32), that is impossible. Hence $x_{g(n)}$ $\{d(g(1)), z_{g(1)}\}$, this implies $x_{g(n)}$ $F(z)$ and n = 1 by (30); then $w_{g(n)}(x_{g(n)}) > w_{g(n)}(z_{g(n)})$ by (31). But $F(z)$ D^k for all k, in particular, $x_{g(n)}$ $D^{\mathrm{st}(n)}$, thus $w_{g(n)}(z_{g(n)})$ $w_{g(n)}(x_{g(n)})$, a contradiction. Therefore, an augmentative sequence is impossible and $z(r)$ FPC.**

2. Let z FPC. We shall construct a scheme r $\hat{A}$ such that $z(r)$ = z. For this purpose it is enough to determine the numberings $n(g)$ for g G_1 and $n_g(i)$ for g G, i J_g. Let us select the numberings $n_g(i)$ so that the following conditions were satisfied:

$$\text{if } w_g(i) > w_g(j), \text{ then } n_g(i) < n_g(j); \tag{55}$$

$$\text{if } i \quad z_g \text{ and } w_g(i) = w_g(z_g), \text{ then } n_g(i) > n_g(z_g). \tag{56}$$

Such numberings obviously exist. The numbering $n(g)$ we shall define with procedure described below. Some *top bargains sequence* **consistent with allocation z will be constructed at each step of this procedure.**

<u>Step 0</u>. At the preliminary step 0 we put $t(g)$ = $t(g, J_g)$ for g G_2 (it follows from (56) that $t(g)$ = z_g). We define D^1, A^1, and F^1 as at the step 0 of the r-

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[3] Some ideas by Roth and Postlewaite (1977) are used in the proof.

algorithm. Put $d(g)$ = 0 if $g \quad G_1$ and $d(g) \quad D^1$ (substantiation is the same as that in the r-algorithm), $J^1 = \{d(g) \mid g \quad G_1\} \setminus \{0\}$.

<u>Step $k+1$</u>. Suppose a top bargains sequence $l_m = \quad g_{m1}, \quad , g_{mn(m)}$ is constructed for each m such that $1 \quad m \quad k$, $G(l_m) = \{g_{ms} \mid 1 \quad s \quad n(m)\} \quad A^m$, $G(l_i) \quad G(l_j) = \quad$ if $i \quad j$, and $t(g) = z_g$ for $g \quad {}_m G(l_m)$. Put $D(l_m) = \{z_g \mid g \quad G(l_m)\} \setminus \{0\}$, $A^{k+1} = G_1 \setminus \quad {}_m G(l_m)$, $D^{k+1} = D^1 \setminus \quad {}_m D(l_m)$, $J^{k+1} = \{d(g) \mid g \quad A^{k+1}\} \setminus \{0\}$, $F^{k+1} = D^{k+1} \setminus J^{k+1}$. Note that F^{k+1} includes the initially vacant dwellings (elements of F^1) not occupied by members of the sequences $l_1, \quad , l_m$, and the dwellings vacated as a result of the described by these sequences moves. Clearly, $z_g \quad D^{k+1}$ is equivalent to $g \quad A^{k+1}$. Assume that $d(g) \quad D^{k+1}$ for all $g \quad A^{k+1}$ (this condition is fulfilled at step 1 and the procedure ensures that it is true for the subsequent steps). Suppose $A^{k+1} \quad$. The following three cases are possible.

(1) $z_g = t(g, D^{k+1}) \quad F^{k+1}$ for some $g \quad A^{k+1}$; put $l_{k+1} = \quad g \quad$.

(2) $t(g, D^{k+1}) \quad F(z)$ for some $g \quad A^{k+1}$; put $l_{k+1} = \quad g \quad$.

(3) Cases (1) and (2) do not happen. Until the sequence l_{k+1} will be constructed, we repeat the following operations. If g_1 is not defined yet, then let us select an arbitrary $g_1 \quad A^{k+1}$. Assume that $g_1, \quad , g_s$ are already defined and agent g_a occupies the dwelling j_a in allocation z ($1 \quad a \quad s$). Put $i = t(g_s, D^{k+1})$; $i \quad F(z)$ since case (2) does not happen.

(3a) $i \quad J^{k+1}$. Then $i = d(g) \quad 0$ for some $g \quad A^{k+1}$. Put $g_{s+1} = g$ if $g \quad \{g_j \mid 1 \quad j \quad s\}$, and put $l_{k+1} = \quad g_m, \quad , g_s \quad$ if $g = g_m$.

(3b) $i \quad J^{k+1}$. Then $i \quad F^{k+1}$ and $i \quad F(z) \quad \{j_s\}$, since cases (1) and (2) do not happen. Thus $i = z_g \quad 0$ for some $g \quad A^{k+1}$, $g \quad g_s$, and $z_g \quad t(g, D^{k+1})$, because case (1) does not happen. Put $l_{k+1} = \quad g_m, \quad , g_s \quad$ if $g = g_m$, and put $g_{s+1} = g$ if $g \quad \{g_j \mid 1 \quad j \quad s\}$.

Eventually, a sequence $l_{k+1} = \quad g_{k+1,1}, \quad , g_{k+1,n(k+1)}$ will be constructed. Put $G(l_{k+1}) = \{g_{k+1,s} \mid 1 \quad s \quad n(k+1)\}$ and $t(g) = t(g, D^{k+1})$ for $g \quad G(l_{k+1})$. By $j_{k+1,a}$ denote the dwelling occupied by agent $g_{k+1,a}$ in allocation z. Let us define an

allocation, x, as follows: $x_g = t(g)$ if $g \quad {}_{m \quad k+1} G(l_m)$, 0 otherwise. It is clear that $x \quad$ DA. The conditions (29) and (30) are satisfied for x with $l = l_{k+1}$ by construction. It follows from the definitions of the sets A^{k+1} and D^{k+1} that $w_g(z_g) \quad w_g(t(g))$ for $g \quad A^{k+1}$. From (55) and (56) we obtain that $t(g) = z_g$ if g $G(l_{k+1})$ and $w_g(z_g) = w_g(t(g))$. Therefore, (31) is true for x with $l = l_{k+1}$. If x and $l = l_{k+1}$ satisfy condition (32), then $z \prec x$, that is impossible since z FPC. Therefore, $w_g(x_g) = w_g(z_g)$ for all $g \quad G(l_{k+1})$. Now from (55) and (56) we obtain $t(g) = z_g$ for $g \quad G(l_{k+1})$. Therefore, case (3b) was not used during the construction of l_{k+1}, $j_{k+1,s} = d(g_{k+1,s+1})$ whenever 1 $\quad s < n(k+1)$, and $j_{k+1,n(k+1)} \quad F^{k+1} \quad \{d(g_{k+1,1})\}$. Thus a top bargains sequence is separated in allocation z. Let us assign the first free numbers in the numbering $n(g)$ to elements of $G(l_{k+1})$, and finish step $k+1$. Since $G(l_k) \quad$, we shall have $A^r =$ for some r, and construction of the numbering $n(g)$ will be completed at step $r \quad 1$. Numberings $n(g)$ and $n_g(i)$ specify the scheme $r \quad \hat{A}$. It is easy to see that $z(r) = z$. Therefore, FPC $\quad E(\hat{A})$. *Q.E.D.*

Theorem 7. If t is the final stage of the EQ-procedure, then $(z(t), p^t)$ is a Walrasian equilibrium for situation $A(t)$.

Proof. The case (3c) happens at the last step of stage t. Cases (2a) and (3a) of the EQ-procedure and the definition of $j(s,g)$ ensure condition (10) for $z(t)$, $p_0 = 0$. Then (Mas-Collel, Whinston, Green, 1995, Lemma 10.B.1, p. 316) condition (7) is fulfilled as well. Take some $g \quad G(t)$ and put $j = j(t, g)$. Suppose $g \quad G_1$. Case (3b) does not happen, thus $u(i, g, t) \quad u(j, g, t)$ for all $i \quad J_g$. Whence $y_{gj}(t) \quad p_j^t \quad y_{gi}(t) \quad p_i^t$. Suppose now $g \quad G_2(t)$; then $j \quad \{0, d(g)\}$. If $j = 0$, then $d(g) \quad \{j(t, g) \mid g \quad G_1\}$; thus $d(g) \quad D_t^1$, and so $y_{gd(g)} \quad p_{d(g)}^t \quad 0 = y_{g0} \quad p_0^t$. If $j = d(g)$, then $j \quad \{j(t, g) \mid g \quad G_1\}$. Thus either $j \quad D_t^1$ or $j \quad F_t^{r(t)}$. Consequently (see case (3a) of the EQ-procedure), $y_{gd(g)} \quad p_{d(g)}^t \quad 0 = y_{g0} \quad p_0^t$. In any case (9) is true for g by Lemma 1. *Q.E.D.*

Lemma 8. If $g \quad G_1$, $j = d(s+1, g)$, and $i \quad I$, then $b_{gi}(s+1) = b_{gi}(s)$ in the case when $j \quad I_1 \quad \{d(s, g)\}$ and $b_{gi}(s+1) = \min\{b_{gi}(s), a_{gj}(s)\}$ otherwise.

Proof. **Put** $d = d(s, g)$, $m = d(s+1, g)$. **From (37) and the definition of** $Db_{gi}(s)$, **by repeating arguments of Theorem 1, we obtain:**

$$b_{gi}(s+1) \quad b_{gm}(s+1) = \min\{b_{gi}(s) \quad b_{gm}(s), w_g(s+1) \quad y_g^0 \quad q(g, m, s+1)\}. \quad (57)$$

It follows from (2) and (57) that

$$b_{gi}(s) \quad b_{gd(s,g)}(s) \quad w_g(s) \quad y_g^0 \quad q(g, d(s,g), s) \quad \text{for all } s. \qquad (58)$$

(1) If $d(s, g) = j$, **then** $w_g(s+1) = w_g(s)$, $m = d$, **and (58) can be written as** $b_{gi}(s) \quad b_{gm}(s) \quad w_g(s+1) \quad y_g^0 \quad q(g, m, s+1)$. **Now (57) implies** $b_{gi}(s+1)$ $b_{gm}(s+1) = b_{gi}(s) \quad b_{gm}(s)$ **for all** i. **But** $b_{g0}(s+1) = b_{g0}(s) = q_{g0}$, **thus** $b_{gi}(s+1) = b_{gi}(s)$ **for all** i.

(2) If $d(s, g) \quad j$, **then** $j = t_{sk}(g)$ **and the EQ-procedure determines** $d(s+1, g)$ **as** $t_{sk}(g)$ **at some step** k **of stage** s. **Then** j **maximizes** $u(i, g, s)$ **over** D_s^k **and** d D_s^k **by Assumption 5 and (36), hence**

$$b_{gd}(s) \quad b_{gj}(s) \quad [p_d^s \quad p_j^s] + [q(g, d, s) \quad q(g, j, s)]. \qquad (59)$$

By summing this inequality with (58), we obtain:

$$b_{gi}(s) \quad b_{gj}(s) \quad w_g(s) \quad y_g^0 + p_d^s \quad p_j^s \quad q(g, j, s). \qquad (60)$$

(2a) $j \quad I_1$. **Then** $m = j$, $w_g(s+1) = w_g(s) + p_d^s \quad p_j^s \quad Q_{gj}(s)$. **Using Assumption 7 and the definition of** $q(g, i, s)$, **let us rewrite (60) in the form:** $b_{gi}(s) \quad b_{gm}(s)$ $w_g(s+1) \quad y_g^0 \quad q_{gj}^2 = w_g(s+1) \quad y_g^0 \quad q(g, m, s+1)$. **Therefrom, using (57), we obtain** $b_{gi}(s+1) = b_{gi}(s)$ **for all** i, **as in case (1).**

(2b) $j \quad I_1$. **Then** $m = 0$, $w_g(s+1) = w_g(s) + p_d^s \quad Q_{gj}(s)$, $a_{gj}(s) = w_g(s+1) \quad y_g^0$, **and (57) takes the form** $b_{gi}(s+1) \quad q_{g0} = \min\{b_{gi}(s) \quad q_{g0}, a_{gj}(s) \quad q_{g0}\}$. *Q.E.D.*

Corollary. $\min\{0, b_{gi}(1)\} \quad b_{gi}(s+1) \quad b_{gi}(s)$ **for** $g \quad G_1$, $i \quad I$, $s \quad 1$.

Proof. **The right-hand inequality follows from Lemma 8. Put** $d = d(s, g)$, $j = d(s+1, g)$, **and assume that** $b_{gi}(s+1) < b_{gi}(s)$. **Then, by Lemma 8,** $d(s, g) \quad j$ **and** $b_{gi}(s+1) = a_{gj}(s) = p_d^s + w_g(s) \quad Q_{gj}(s) \quad y_g^0$, **while** $j = t_{sk}(g)$ **for some step** k **of stage** s. **By (59),** $b_{gj}(s) \quad b_{gd}(s) \quad p_j^s \quad p_d^s + q(g, j, s) \quad q(g, d, s)$. **Then**

(58) with $i = j$ implies $w_g(s) \quad y_g^{\mathbf{0}} \quad q(g, d, s) \quad p_j^s \quad p_d^s + q(g, j, s) \quad q(g, d, s)$, hence $0 \quad w_g(s) + p_d^s \quad p_j^s \quad q(g, j, s) \quad y_g^{\mathbf{0}} \quad w_g(s) + p_d^s \quad Q_{gj}(s) \quad y_g^{\mathbf{0}} = b_{gi}(s+1)$. Therefore, $b_{gi}(s+1) \quad 0$ if $b_{gi}(s+1) \quad b_{gi}(1)$. *Q.E.D.*

Lemma 9. Suppose $g \quad G$, $i = j(s, g)$ and case (3) holds at the last step of stage s. Then $b_{gi}(s) \quad q(g, i, s) \quad p_i^s$. If, additionally, $g \quad G_1$, then $y_{gi}(s, p^s) \quad y_g^{\mathbf{0}}$.

Proof. Suppose $g \quad G_2(s)$, then $y_{gi}(s) = y_{gi}$. The statement of the lemma is obvious with $i = 0$ ($b_{g0} = q_{g0} = p_0^s = 0$). If $i \quad 0$, then $i = d(g)$; in this case either $i \quad D_s^1$ (and $p_i^s < y_{gi}$ by the definition of D_s^1) or $i \quad D_s^1 \setminus \{j(s, h) \mid h \quad G_1\}$ (and $p_i^s \quad y_{gi}$ by case (3a) of the EQ-procedure). Suppose now $g \quad G_1$. Then $b_{g0}(s) = q_{g0}$ by (37), $y_{g0} = 0$, $0 \quad D_s^k$, and i maximizes $u(i, g, s)$ over D_s^k. From $u(i, g, s) \quad u(0, g, s)$ and (33) we obtain $y_{gi}(s) \quad p_i^s \quad 0$. So, $b_{gi}(s) \quad q(g, i, s) \quad p_i^s$ in any case.

The condition $y_{gi}(p^s) \quad y_g^{\mathbf{0}}$ for $s = 1$ is true by Lemma 2. Suppose this condition is true for $s \quad 1$ ($s \quad 2$) and let us prove it for s. By the definition, i maximizes $u(i, g, s)$ over D_s^k for some k. If $d(s \quad 1, g) = d(s, g)$, then $d(s \quad 1, g) = d(s, g) = d$. Thus $d \quad D_s^k$ and, therefore, $u(i, g, s) \quad u(d, g, s)$, whence $y_{gi}(s) \quad y_{gd}(s) \quad p_i^s \quad p_d^s$. From this, using (58), we obtain: $y_{gi}(s, p^s) = w_g(s) \quad p_i^s + p_d^s \quad q(g, i, s) \quad b_{gi}(s) \quad b_{gd}(s) + q(g, d, s) + y_g^{\mathbf{0}} \quad p_i^s + p_d^s \quad q(g, i, s) = [y_{gi}(s) \quad y_{gd}(s)] \quad [p_i^s \quad p_d^s] + y_g^{\mathbf{0}} \quad y_g^{\mathbf{0}}$. Assume now that $d(s \quad 1, g) \quad d(s, g)$. It is easy to see that $y_{gi}(s, p)$ is a strictly decreasing function of $b_i(s, p)$. Therefore, $y_{gi}(s, p^s) < y_g^{\mathbf{0}}$ would imply $b_i(s, p^s) = q(g, i, s) \quad q(g, d(s,g), s) + p_i^s \quad p_{d(s,g)}^s > Db_{gi}(s) = b_{gi}(s) \quad b_{gd(s,g)}(s)$ and thus $y_{gi} \quad y_{gd(s,g)} < p_i^s \quad p_{d(s,g)}^s$ contrary to (59). *Q.E.D.*

Lemma 10. The sets Al and Pr are finite.

Proof. If case (3) of the EQ-procedure happens at step $r(s)$ and $i = j(s, g)$, then $p_i^{s+1} \quad p_i^s + d \quad y_{gi}(s) + d \quad b_{gi}(s) + d \quad b_{gi}(1) + d$ by Lemma 9 and Corollary from Lemma 8. If case (3) of the EQ-procedure does not happen at step $r(s)$ or $i \; \{d(s+1, g) \mid g \; G\}$, then $p_i^{s+1} \; p_i^s$.

It is easy to see that case (3) will necessarily happen at some stage; let m be the number of the first such stage. Put $K = \max\{\max\{b_{gi} \mid g \; G, i \; J_g\} + d, \max\{p_i^s \mid i \; I, s < m\}\}$. It is clear that $0 \; p_i^s \; K$ for all i, s. Since p_i^{s+1} $\{p_i^s, p_i^s + d, p_i^s \;\;\;, p_i^0\}$, Pr is finite. Finiteness of Al is obvious. *Q.E.D.*

Lemma 11. If the bargains corresponding to a sequence $l = g_1, \;\; , g_n$ are carried out at stage s of the EQ-procedure and $n > 1$, then $u(d(s,g), g, s) < u(d(s+1,g), g, s)$ for some $g \; \{g_1, \;\; , g_n\}$.

Proof. If l is a chain then $u(d(s,g_1), g_1, s) < u(d(s+1,g_1), g_1, s)$ by the definition of correct sequence. Let l be a cycle. Sequence l does not appear before stage s (otherwise it would be carried out). Assume that l appears for the first time at step k of stage s; $t_{sk}(g_j) \; d(s, g_j)$ for all j (as $n > 1$) and $t_{sk}(g_m)$ is not equal to $L(g_m)$ for some m at the beginning of step k (otherwise this cycle would appear earlier). Then, for $g = g_m$, we have $u(d(s,g), g, s) < u(t_{sk}(g), g, s)$ and $d(s+1, g) = t_{sk}(g)$. *Q.E.D.*

Lemma 12. There exists an integer M such that either the EQ-procedure will be finished before stage $M+1$, or $b_{gi}(s) \; b_{gi}(s+1) \; d > 0$ for some i, $g \; G_1$, and $s \; M$.

Proof. If $i \; D_s^1$ for all s then the values p_i^s do not decrease with respect to s. Suppose $i \; D_s^1$ for some s and let $s(i)$ denote the smallest such s. Take some $t > s(i)$. The definition of D_s^1 implies that $p_i^{s(i)} \; p_i^0$; then $p_i^t \; p_i^0$ too (since all p_i^s are multiples of d), and thus $i \; D_t^1$. Case (3b) of the EQ-procedure assures us that

$$s \; s(i) \text{ and } p_i^s < p_i^{s+1} \text{ imply } i \; \{d(s+1, g) \mid g \; G_1\}. \tag{61}$$

Put $B_s = \{d(s, g) \mid g \; G_1\}$ (the set of the dwellings occupied by consumers at the beginning of stage s). If $p_i^t > p_i^{t+1}$ and $i \; B_{t+1}$, then $i \; B_s$ for all $s \; t$ (if dwelling i becomes tenantless at some stage s, then $p_i^s = p_i^{\mathbf{0}}$ by the definition of a correct sequence, and this price will not change while the dwelling is occupied); hence, taking into account (61), $s(i) < s \quad t$ implies $p_i^{s\ \mathbf{1}} \quad p_i^s$. Therefore, both conditions $p_i^t > p_i^{t+\mathbf{1}}$ and $i \quad B_{t+\mathbf{1}}$ can be true only for finitely many stages t. Then there exists $s_\mathbf{0} \quad \max\{s(i)\}$ such that

$$s > s_\mathbf{0} \text{ and } p_i^s > p_i^{s+\mathbf{1}} \text{ imply } i \quad B_{s+\mathbf{1}}. \qquad \textbf{(62)}$$

The conditions (61) and (62) mean that prices of the dwellings belonging to $I \setminus B_{s+\mathbf{1}}$ cannot be changed at stages $s > s_\mathbf{0}$. Taking into account Lemma 10, put M = s_0 + |Al|×|Pr| +1.

Assume that the EQ-procedure was not ended in |Al|×|Pr| stages after the stage $s_\mathbf{0}$. Then there exist some stages a and b, $s_\mathbf{0} < a < b$, such that $p^a = p^b$ and $z^a = z^b = z$ (the initial allocations, as well as prices, coincide at the stages a and b). The procedure is cyclic starting with stage a.

Assume that the allocation is invariable within the stages $s \quad [a, b)$, $z^s = z$. Then $B_s = B_a$ for $s \quad [a, b)$ and, taking into account Lemma 8,

$$b_{gi}(s) = b_{gi}(a) \quad \text{for } s \quad [a, b) \text{ and all } g, i. \qquad \textbf{(63)}$$

If s is not the last stage of the EQ-procedure, then either $z^s \quad z^{s+\mathbf{1}}$ or $p^s \quad p^{s+\mathbf{1}}$. Thus $p^s \quad p^{s+\mathbf{1}}$ for $s \quad [a, b)$. Let us consider a dwelling, i, such that $i = z_g$ for $g \quad G_\mathbf{1}$, g participates in a maximal feasible non-correct sequence at some stage $m \quad a$, and after that i rises in price at a stage n: $p_i^m = p_i^n < p_i^{n+\mathbf{1}}$. The agent g, possibly, participates in feasible non-correct sequences at some stages $s \quad [m, n)$; assume that prices of the dwellings occupied by members of such sequences do not increase in this interval. Then g will be the first member of a maximal feasible non-correct sequence at some step of stage n; this is impossible since, for some $h \quad G_\mathbf{1}$, $u(i, h, m) > u(d(n,h), h, n)$ and $i = t_n(h, I)$. Therefore, $i(\mathbf{1})$, $m(\mathbf{1})$, and $n(\mathbf{1})$ exist such that $p_{i(\mathbf{1})}^{m(\mathbf{1})} = p_{i(\mathbf{1})}^{n(\mathbf{1})} < p_{i(\mathbf{1})}^{n(\mathbf{1})+\mathbf{1}}$, $[m(\mathbf{1}), n(\mathbf{1})) \quad [m, n)$, $i(\mathbf{1}) = z_{h(\mathbf{1})}$, and agent $h(\mathbf{1})$ is a member of a maximal feasible non-correct sequence at stage $m(\mathbf{1})$.

It is clear that there exist i, m, and n such that $a \le m < n$ and $p_i^m > p_i^{m+1} = p_i^n < p_i^{n+1}$ (the price of dwelling i decreases at stage m and increases at stage n). Then $i = z_g$ and g participates in a maximal feasible non-correct sequence at stage m. As we have proved above, some stages $m(1)$ and $n(1)$ can be found such that $[m(1), n(1)) \subset [m(1), n(1))$, the price of some dwelling $i = z_{h(1)}$ is invariable in the interval $[m(1), n(1))$ and increases at stage $n(1)$, and agent $h(1)$ participates in some feasible non-correct sequence at stage $m(1)$. Repeating this reasoning, we shall construct the infinite sequence of intervals $[m(k), n(k))$ with integer extremities and monotonously decreasing lengths, that is impossible. The contradiction is caused by the assumption $z^s = z$ for all $s \in [a, b)$. Therefore, some non-trivial sequences of bargains are actualized at stages $s \in [a, b)$.

By the rule of selecting the dwelling $t_{sk}(g)$ we have $u(d(s+1, g), g, s) \ge u(d(s, g), g, s)$ for all g and s. It follows from (33) and Lemma 11 that

$$p^s_{d(s,\, g)} - p^s_{d(s+1,\, g)} \ge y_{gd(s,g)}(s) - y_{gd(s+1,g)}(s) \tag{64}$$

with strict inequality for at least one pair (g, s), $g \in G_1$, $s \in [a, b)$. Summing inequalities (64) over s from a to $b - 1$, we get:

$$\sum_{s=a}^{b-1} (p^s_{d(s,\, g)} - p^s_{d(s+1,\, g)}) \ge \sum_{s=a}^{b-1} [y_{gd(s,g)}(s) - y_{gd(s+1,g)}(s)]. \tag{65}$$

Since $p^a_{d(a,g)} = p^b_{d(b,g)}$, we have $\sum_{s=a}^{b-1} (p^s_{s,} - p^s_{d(s+1,\, g)}) = \sum_{a+1}^{1} (p_{d(s\ g)} - p_{d(s\ g)}) + p_{a,g)} - p^{b-1}_{d(b,g)} = \sum_{s=a+1}^{b} (p^s_{d(s,g)} - p^{s-1}_{d(s,g)})$. By Corollary from Lemma 8, using $d(a, g) = d(b, g)$, we get $y_{gd(a,g)}(a) - y_{gd(b,g)}(b - 1) = y_{gd(b,g)}(a) - y_{gd(b,g)}(b - 1) \ge y_{gd(b,g)}(b) - y_{gd(b,g)}(b - 1)$. Then $\sum_{s=a}^{b-1} [y_{gd(s,g)}(s) - y_{gd(s+1,g)}(s)] \ge \sum_{s=a+1}^{b} [y_{gd(s,g)}(s) - y_{gd(s,g)}(s - 1)]$. If $i = d(s, g)$, $j = d(s - 1, g)$, and $g \in G_1$, then $q(g, i, s) = q_{gi}(i) = q^2_{gi}$ and $q(g, i, s - 1) = q_{gi}(j) = q^2_{gi} + Q_{gi}(s - 1)$. Let us put $R_{gs} = y_{gd(s,g)}(s) - y_{gd(s,g)}(s - 1) = b_{gd(s,g)}(s) - b_{gd(s,g)}(s - 1) + Q_{gd(s,g)}(s - 1)$. It follows from (65) that $\sum_{s=a+1}^{b} R_{gs} \le \sum_{s=a+1}^{b} (p^s_{d(s,\, g)} - p^{s-1}_{d(s,\, g)})$ with strict inequality for at least one g. Summing over $g \in G_1$, we get:

$$\sum_g \sum_{s=a+1}^{b} (p^s_{d(s,\, g)} - p^{s-1}_{d(s,\, g)}) > \sum_g \sum_{s=a+1}^{b} R_{gs}. \tag{66}$$

(1) If R_{gs} 0 for all g and s, then $_g\ {}^{b}_{s=a+1}(p^{s}_{d(s,g)}\ \ p^{s\ 1}_{d(s,g)}) > 0$. In the last sum, the changes in prices are summed for all dwellings of the set B_s at stage s 1. But (61) and (62) imply that the prices of dwellings outside this set are invariable at stage s 1 if $s > a$. Thus $_i(p^b_i\ \ p^a_i) =\ _g\ {}^{b}_{s=a+1}(p^{s}_{d(s,g)}$ $p^{s\ 1}_{d(s,g)}) > 0$ contrary to $p^b = p^a$.

(2) Suppose $R_{gs} < 0$ for some g G_1 and s $[a+1, b]$. Put $i = d(s, g)$. From $R_{gs} < 0$ we get $b_{gi}(s\ \ 1)\ \ b_{gi}(s) > Q_{gi}(s\ \ 1)$ 0. Assumption 8 and EQ-procedure imply that p^s_i, $w_g(s)$, $b_{gi}(s)$, and $Q_{gi}(s\ \ 1)$ are rationals and also they are multiples of d. Now it is clear that $b_{gi}(s\ \ 1)\ \ b_{gi}(s)\ \ d > 0$. *Q.E.D.*

Theorem 8. The EQ-procedure is finite under Assumption 8.

Proof. If the EQ-procedure was not ended in M stages, then $b_{gi}(s)\ \ b_{gi}(s+1)$ $d > 0$ for some g G_1, i I, s M (Lemma 12); d does not depend on g, i, s. Let us apply Lemma 12 to the final situation of stage M. If the procedure will not terminate in the following M stages, then some reservation price will decrease at least by d within these M stages. The reservation prices do not increase and are bounded below by Corollary from Lemma 8; thus the procedure is finite. *Q.E.D.*

Lemma 13. For any Walrasian equilibrium (z, p), some normal equilibrium (z^0, p^0) exists such that $z(z^0) = z(z)$.

Proof. Let (z, p) be an equilibrium. By Lemma 3, there exists a vector, p^1, such that (z, p^1) is an equilibrium and $c(p^1_{d(g)})$ $y_{gd(g)}$ for all g G_2. Put $p^1 = c(p^1)$. Assume that equilibrium (z, p^1) is not a normal one. Then there exist some i and j such that i j and $p^1_i > p^1_j$ 0. If i $\{j(z, g) \mid g$ $G_1\}$, then $g(i)$ G_2 since $p^1_i > 0$. It follows from i j that $g(j)$ G_2 and $p^1_i > p^1_j$ $y_{g(j)j} = y_{g(i)i}$, contrary to Lemma 2. Therefore, $i = j(z, h)$ for some h G_1. If $d(h)$ $\{i, j\}$, then $(y_{hj}\ \ y_{hi})$ $(p^1_j\ \ p^1_i) = p^1_i\ \ p^1_j > 0$. If $d(h) = j$, then $(y_{hj}\ \ y_{hi})$ $(p^1_j\ \ p^1_i) = (b_{hj}$ $b_{hi})$ $(q^2_{hj}\ \ r\times q^1_{hi}\ \ q^2_{hi})$ $(p^1_j\ \ p^1_i) = r\times q^1_{hi} + p^1_j\ \ p^1_i > 0$. In both cases we have a contradiction with Lemma 1. Therefore, $j(z, h) = d(h) = i$.

Let us define $p^2 \quad P$: $p_i^2 = p_j^1$, $p_k^2 = p_k^1$ for $k \quad i$. Let p^2 be a unique price vector satisfying the condition $c(p^2) = p^2$, and define $z^2 \quad FD$ as follows: z_g^2 = $(y_{gj(z,g)}(p^2), j(z,g))$ for all $g \quad G$. Let us show that (z^2, p^2) is an equilibrium.

Condition (10) is fulfilled for (z^2, p^2) because it is fulfilled for (z, p^1) and $p^2 \quad p^1$. If $k \quad i$, then, applying Lemma 1 to equilibrium (z, p^1), we obtain $p_i^2 \quad p_k^2 <$ $p_i^1 \quad p_k^1 \quad y_{hi} \quad y_{hk}$. Consider $g \quad G_1 \setminus \{h\}$. Then $j(z, g) \quad i$ and $d(g) \quad i$. If $k \quad i$, then, applying Lemma 1 to equilibrium (z, p^1), we obtain $p^2_{j(z,g)} \quad p_k^2 = p^1_{j(z,g)}$ $p_k^1 \quad y_{gj(z,g)} \quad y_{gk}$. It follows from $i \quad j$ that $y_{gi} = y_{gj}$ if $d(g) \quad j$, and $y_{gi} \quad y_{gj}$ if $d(g) = j$. Thus $p^2_{j(z,g)} \quad p_i^2 = p^1_{j(z,g)} \quad p_j^1 \quad y_{gj(z,g)} \quad y_{gj} \quad y_{gj(z,g)} \quad y_{gi}$. So, condition (9) with $g \quad G_1$ is true for (z^2, p^2) (Lemma 1). If $g \quad G_2$ and $d(g) \quad i$, then (9) is fulfilled for g since $p_k^1 = p_k^2$ for all $k \quad J_g$. If $g \quad G_2$ and $d(g) = i$, then $g(j) \quad G_2$ and $y_{gi} = y_{g(j)j} \quad p_j^1 = p_i^2 < p_i^1$; thus, $j(z, g) = j(z^2, g) = 0$ and $u_g(z^2) \quad v_{gi}(p^2) =$ $p_i^2 \quad y_{gi} \quad 0$. Now (9) is proved for all $g \quad G$. Hence, (z^2, p^2) is an equilibrium.

Note that $z(z^2) = z(z)$ and $p_i^2 = p_j^2$. It is clear that, proceeding repeatedly, we shall construct the desired normal equilibrium. *Q.E.D.*

Lemma 14. Suppose (z, p) is a normal equilibrium and $g \quad h$. If $\{g, h\} \quad G_1$ and $d(g) \quad d(h)$, then $u_g(z) = u_h(z)$; if $\{g, h\} \quad G_2$, then $u_g(z) \quad u_h(z) = w_g \quad w_h$.

Proof. Suppose (z, p) is a normal equilibrium, $p = c(p)$, and $g \quad h$. Assume that $\{g, h\} \quad G_1$ and $d(g) \quad d(h)$. Then $p_{d(g)} = p_{d(h)}$ and $p_{d(g)} = p_{d(h)}$. Put $i = j(z, g)$. If $i \quad \{d(g), d(h)\}$, then $u_g(z) = w_g + p_{d(g)} \quad p_i + y_{gi} = w_h + p_{d(h)} \quad p_i + y_{hi} =$ $v_{hi}(p) \quad u_h(z)$ by (9). If $i \quad \{d(g), d(h)\}$, then $b_{gi} \quad q_{gi}^2 = b_{hi} \quad q_{hi}^2 = b_{hd(h)} \quad q^2_{hd(h)}$. Recall that $q^1_{gd(g)} = q^1_{hd(h)} = 0$. Thus, if $i = d(g)$, then $u_g(z) = v_{gd(g)}(p) = v_{hd(h)}(p)$ $u_h(z)$. Also, if $i = d(h)$, then $u_g(z) = w_g + p_{d(g)} \quad p_i + b_{gi} \quad q_{gi} \quad w_g + p_{d(h)} \quad p_i + b_{hi}$ $q_{hi}^2 = v_{hi}(p) \quad u_h(z)$. Hence, $u_g(z) \quad u_h(z)$ in any case. But $g \quad h$ implies $h \quad g$, thus $u_g(z) = u_h(z)$. Assume now that $\{g, h\} \quad G_2$. Then $d(g) \quad d(h)$, $p_{d(g)} = p_{d(h)}$, and, using Lemma 2, we obtain $u_g(z) \quad w_g = \max\{y_{gd(g)}, p_{d(g)}\} = \max\{y_{hd(h)}, p_{d(h)}\} = u_h(z) \quad w_h$. *Q.E.D.*

Theorem 9. For any normal equilibrium (z, p), some standard equilibrium (z^0, p) exists such that $j(z^0, g) \quad j(z, g)$ for all $g \quad G_1$.

Proof. Suppose (z, p) is a normal equilibrium (it exists by Lemma 13), and put $p = c(p)$. If (z, p) is not a standard equilibrium, then $d(g) \quad j(z, g) \quad d(g)$ for some $g \quad G_1$. Let us take such g and put $j = j(z, g)$, $i = d(g)$. Then $i \quad 0$, $j \quad 0$, $p_i = p_j$, $b_{gi} \quad q^2_{gi} = b_{gj} \quad q^2_{gj}$, and $u_g(z) = w_g + p_{d(g)} \quad p_j + b_{gj} \quad q_{gj} \quad w_g + p_{d(g)} \quad p_i + b_{gi} \quad q^2_{gi} = v_{gi}(p)$. Taking into account (9), we obtain $u_g(z) = v_{gi}(p)$ (the dwellings i and j are equivalent to agent g). Put $k(g) = i$. If $i = j(z, h)$ for some $h \quad G_2$, then $i = d(h)$ (since $J_h = \{d(h), 0\}$ and $i \quad 0$) and $p_i \quad y_{hi}$ (Lemma 2). It follows from $i \quad j$ that $g(j) \quad G_2$ and $p_j = p_i \quad y_{hi} = y_{g(j)j}$. But $j(z, g) = j$ implies $p_j \quad y_{g(j)j}$ (Lemma 2). Therefore, $p_i = y_{hi}$ (zero and i are equivalent to agent h). Put $k(h) = 0$. If $i = j(z, h)$ for some $h \quad G_1$, then (8) and Assumption 1 imply $h \quad g$ and $d(h) \quad i$ (as $i \quad 0$). If $d(h) = j \quad i = j(z, h)$, then, as we have proved above, $u_h(z) = v_{hj}(p)$. If $d(h) \quad j$, then $d(h) \quad \{i, j\}$ and $y_{hi} = y_{hj}$ (as $i \quad j$); thus $u_h(z) \quad v_{hj}(p) = p_j \quad p_i + y_{hi} \quad y_{hj} = 0$. In any case, the dwellings i and j are equivalent to agent h. Put $k(h) = j$. At last, put $k(f) = j(z, f)$ for all $f \quad G$ such that $f \quad g$ and $j(z, f) \quad i$. Obviously, $k(h) \quad j(z, h)$ for all $h \quad G_1$. Put $z^1 = (z^1(h) \mid h \quad G)$, where $z^1(h) = (y_{hk(h)}(p), k(h))$. It is easy to see that (z^1, p) is a normal equilibrium, and, in this equilibrium, the number of agents violating the condition for a normal equilibrium be standard is less than that in z. Proceeding repeatedly, we shall construct some standard equilibrium (z^0, p). *Q.E.D.*

Lemma 15. If $h \quad GS$, then $I(t(h)) = \{d(g) \mid g \quad G(h, 0)\}$.

Proof. Suppose $h \quad GS$. Then all dwellings owned by suppliers of group h are of type $t(h)$. And conversely, if $i \quad I(t(h))$, then i is owned by a supplier of group h (see the definition of dwelling types). *Q.E.D.*

Lemma 16. The quantities a_{nkh} are well defined.

Proof. Suppose a_{nkh} is defined, $\{i,j\} \subseteq I(k)$ and $\{x,y\} \subseteq G(h,n)$; then $x \neq y$, $i \neq j$, and $\{d(x), d(y)\} \subseteq I(n)$. It follows from $x \neq y$ that either $\{x,y\} \subseteq G_1$ or $\{x,y\} \subseteq G_2$. Assume that $\{x,y\} \subseteq G_1$. If $k \neq n$, then $\{i,j\} \cap \{d(x),d(y)\} = \emptyset$; thus $x \neq y$ and $i \neq j$ imply $y_{xi} = y_{yi} = y_{yj}$. Whence, a_{nkh} is well defined. If $n = k$, then $x \neq y$ and $d(x) \neq d(y)$ imply $y_{xd(x)} = b_{xd(x)} - q^2_{xd(x)} = b_{xd(y)} - q^2_{xd(y)} = b_{yd(y)} - q^2_{xd(y)} = y_{yd(y)}$, thus a_{nnh} is well defined. Assume now that $\{x,y\} \subseteq G_2$. Then $y_{x0} = y_{y0} = 0$ by the definition, and $y_{xd(x)} = y_{yd(y)}$ since $d(x) \neq d(y)$. Thus a_{nkh} is well defined for $k \in \{0, t(h)\}$. *Q.E.D.*

Lemma 17. If $h \in GS$ and X is a feasible solution of the AT problem, then $\sum_{m}\sum_{a \in GC} X_{mt(h)a} \le X_{00h}$.

Proof. Suppose $h \in GS$ and X is a feasible solution of the AT problem. Clearly, $\bigcup_{g \in G(h,0)} J_g = \{d(g) \mid g \in G(h,0)\} \subseteq \{0\}$. By Lemma 15, variables X_{0kh} are defined only for $k \in \{t(h), 0\}$. If $k = t(h)$ and $n = 0$, then the conditions (39) and (38) take the form: $\sum_{m}\sum_{a \in GC} X_{mt(h)a} + X_{0t(h)h} \le C_{t(h)}$ and $X_{0t(h)h} + X_{00h} = D_{h0}$. The statement of the lemma follows now from $C_{t(h)} = D_{h0}$ (Lemma 15). *Q.E.D.*

Theorem 10.[4] Suppose (z,p) is a standard equilibrium, $x = x(z)$, $X_{nkh} = \sum_{g \in G(h,n)}\sum_{i \in I(k)} x_{gi}$ for $(n,k,h) \in U$, $g_{hn} = a_g(z,p)$ for $g \in G(h,n)$, and $p_k = c(p_i)$ for $i \in I(k)$; then X and (g, p) are optimal solutions of the AT and AT* problems respectively. If X is an integral optimal solution of AT (in particular, a basic one), (g,p) is an optimal solution of AT*, $\bar{p}_i = p_k$ for $i \in I(k)$, and $\bar{p} = (\bar{p}_i \mid i \in I)$, then there exists some basic optimal solution x of T such that $(z(x,\bar{p}), r(\bar{p}))$ is a standard equilibrium and $X_{nkh} = \sum_{g \in G(h,n)}\sum_{i \in I(k)} x_{gi}$.

Proof. (1) Suppose (z, p) is a standard equilibrium and $x = x(z)$. It follows from the definition of a normal equilibrium and Lemma 11 that g and p in the theorem formulation are well defined. Obviously, X is a feasible solution to AT if $X_{nkh} = \sum_{g \in G(h,n)}\sum_{i \in I(k)} x_{gi}$. By Theorem 2, $(a(z,p), c(p))$ is an optimal solution of T*, whence (g,p) is a feasible solution of AT*. Since (z,p) is a standard

[4] The formulation and proof use the definitions introduced in section 2.1.

equilibrium, $i \quad d(g)$ and $i \quad d(g)$ imply $x_{gi} = 0$. Thus $\quad_{n,k,h} a_{nkh} X_{nkh} = \quad_{g,i} y_{gi} x_{gi}$ $= \quad_{g} a_g(z, p) + \quad_{i} c(p_i) = \quad_{h,n} \quad_{g} \quad_{G(h,n)} a_g(z, p) + \quad_{k} \quad_{i} \quad_{I(k)} c(p_i) = \quad_{h,n} D_{hn} g_{gn} +$ $\quad_{k} C_k p_k$. Consequently, X and (g, p) are optimal solutions to AT and AT* respectively.

(2) Note that the AT problem is totally unimodular, therefore all its basic feasible solutions are integral. Suppose X is an integral optimal solution to AT and (g, p) is an optimal solution to AT*. Let us describe the procedure constructing a vector x. In the beginning, all components x_{gi} of x are equal to zero. Every time when we shall set $x_{gi} = 1$, we shall mean that $x_{gk} = 0$ for all $k \quad J_g \setminus \{i\}$, and say that the vector $x_g = (x_{gk} \mid k \quad J_g)$ is *determined* and dwelling i (if $i \quad 0$) is *allocated* now. For each pair (h, n) such that $G(h, n)$ and $h \quad$ GC, let us select X_{nnh} representatives in $G(h, n)$ so that x_g was not still determined for each chosen agent g (the choice is possible since $X_{nnh} \quad D_{hn}$). Put $x_{gd(g)} = 1$ for each chosen agent g. If $n = t(b)$ for some $b \quad$ GS, then the dwellings initially occupied by chosen agents are owned by suppliers of group b; for each such supplier g, vector x_g was not still determined, and we put $x_{g0} = 1$.

The further construction is carried out step by step. For $(n, k, h) \quad U$, let us denote by X^s_{nkh} the number of all (n, h)-agents g such that we had put $x_{gi} = 1$ with $i \quad I(k)$ before step s. Clearly, X^s_{nkh} is the sum over $g \quad G(h, n)$ and $i \quad I(k)$ of components x_{gi} of all vectors x_g determined before step s. Put $I^s(k) = \quad_{n,h} X^s_{nkh}$ (if $k \quad 0$, then $I^s(k)$ is the number of dwelling in $I(k)$ allocated before step s). It follows from the above construction that: $X^1_{nkh} = X_{nkh}$ if $k = n$ and $h \quad$ GC; $X^1_{nkh} = \quad_{a \quad GC} X_{t(h)t(h)a} \quad X_{00h}$ if $n = k = 0$ and $h \quad$ GS (Lemma 17); $X^1_{nkh} = 0$ otherwise. Besides, $I^1(k) = \quad_{h \quad GC} X_{kkh} \quad C_k$ by (39) if $k \quad 0$, and $I^1(0) = \quad_{h \quad GC} X_{00h} + \quad_{b \quad GS} \quad_{h \quad GC} X_{t(b)t(b)h} \quad \quad_{h} X_{00h} \quad C_0$ by Lemma 17 and (39). Let us describe a step $s \quad 1$ now.

Assume that the following *initial conditions* hold before step s: if $k \quad 0$ or $h \quad$ GC, then $X^s_{nkh} \quad \{X_{nkh}, 0\}$; if $h \quad$ GS and $k \quad \{t(h), 0\}$, then $X^s_{0kh} \quad X_{0kh}$; $I^s(k) \quad C_k$

for all k. These conditions are true before step 1. The following cases are possible.

(a) X^s_{nkh} = 0 for some triplet (n, k, h) U with h GC. Then k n. Let us select X_{nkh} representatives in $G(h, n)$ such that x_g is not still determined for each chosen agent g (the choice is possible since $_{m \quad k} X^s_{nmh} + X_{nkh} \quad _m X_{nmh} = D_{hn}$). Select also X_{nkh} unallocated dwellings in $I(k)$ (the choice is possible since $I^s(k) + X_{nkh} = \ _{(m,a)\quad(n,h)} X^s_{mka} + X_{nkh} \quad _{m,a} X_{mka} \quad C_k$). Let us establish an arbitrary bijection $i(g)$ between the set of chosen agents and that of chosen dwellings. Put $x_{gi(g)}$ = 1 for all g. If $k = t(b)$ for some b GS, then the dwellings being allocated at step s are owned by suppliers of group b; for each such supplier g, vector x_g was not still determined (by construction, we determine x_g for g G_2 only after $d(g)$ was allocated), and we put x_{g0} = 1. Step s is completed. Note that the initial conditions are true for s+1. In particular, $I^{s+1}(0) =$

$$_{b \ \mathrm{GS} \quad n \quad h \ \mathrm{GC}} X^{s+1}_{nt(b)h} + \ _{n \quad h \ \mathrm{GC}} X^{s+1}_{n0h} \quad _{n \quad h \ \mathrm{GC}} X_{n0h} + \ _{b \ \mathrm{GS} \quad n \quad h \ \mathrm{GC}} X_{nt(b)h}$$

$_{n,h} X_{n0h} \quad C_0$ by Lemma 17 and (39), $X^{s+1}_{00h} = \ _{n \quad a \ \mathrm{GC}} X^{s+1}_{nt(h)a} \quad _{n \quad a \ \mathrm{GC}} X_{nt(h)a}$ X_{00h} for h GS by Lemma 17.

(b) Case (a) does not happen and $X^s_{00h} < X_{00h}$ for some h GS. Then case (c) did not happen before step s and thus $X^s_{0t(h)h}$ = 0. Let us select $X_{00h} \quad X^s_{00h}$ suppliers in group h such that x_g was not still determined for each chosen agent g (the choice is possible since $D_{h0} \quad X^s_{00h} \quad X_{00h} \quad X^s_{00h}$). Put x_{g0} = 1 for each chosen agent g. Step s is completed. The initial conditions for s+1 are obviously true.

(c) The preceding cases do not hold and $X^s_{nkh} < X_{nkh}$ for some triplet (n, k, h) U. Then $(n, k, h) = (0, t(h), h)$ for some h GS, $X^s_{00h} = X_{00h}$ and $X^s_{0t(h)h}$ = 0. $D_{h0} = X_{00h} + X_{0t(h)h}$ by (38), thus it is possible to select $X_{0t(h)h}$ suppliers in group h such that, for each chosen agent g, x_g was not still determined and, consequently, dwelling $d(g)$ was not still allocated. Put $x_{gd(g)}$ = 1 for each chosen agent g. Step s is completed and the initial conditions for

s**+1 are true.**

(d) $X^{s}_{nkh} = X_{nkh}$ **for all (**n, k, h**)** U**. The construction is completed.**

After finitely many steps, the construction will be completed, vectors x_g **will be determined for all** g**, and a feasible solution** x **to problem T will be constructed. Surely,** $_{g\ G(h,n)}\ _{i\ I(k)} x_{gi} = X_{nkh}$ **for all (**n, k, h**)** U**. Put** $a_g = g_{hn}$ **for** g $G(h, n)$ **and put** $\bar{p}_i = p_k$ **for** i $I(k)$**. Obviously, (**$a, \bar{p}$ **) is a feasible solution to T*.** X **and (**g, p**) are optimal solutions to T and T* respectively, then** $_{g,i}\, y_{gi} x_{gi}$ $=$ $_{n,k,h}\, a_{nkh} X_{nkh}$ $=$ $_{h,n}\, D_{hn} g_{hn}$ $+$ $_{k}\, C_k p_k$ $=$ $_{g}\, a_g$ $+$ $_{i}\, \bar{p}_i$**; thus** x **and (**$a, \bar{p}$ **) are optimal solutions to T and T* respectively. Then** x **is a basic (because it is integral) optimal solution to T. Theorem 2 and the definitions of** x **and** $\bar{p}$ **imply that (**$z(x, \bar{p})$**,** $r(\bar{p})$**) is a standard equilibrium.** *Q.E.D.*

Theorem 11. **max{**p_j **|** p $P(A_2)$**}** y_{fj} d **min{**p_j **|** p $P(A_1)$**}** y_{fj}**.**

*Proof***. If** $c(p) = p$ $P(A)$**, then (**z, p**) with some** z **is a standard equilibrium for situation A. Then (**g, p**) with some** g **is an optimal solution to AT* (Theorem 10). Note that** $d = F_0(A_2)$ $F_0(A_1)$ w_f $y_{fj} = f_0(A_2)$ $f_0(A_1)$ y_{fj}**. Let us write AT*(**k**), k {1, 2}, for the AT* problem in situation** A_k **and denote by** $F_k(g, p)$ **the objective function in the AT*(**k**) problem. It is clear that the AT*(**k**) problems have the same set of feasible solutions** D**. Suppose** p^k $P(A_k)$ **and (**g^k, p^k**) is an optimal solution to the AT*(**k**) problem. Then** $f_0(A_2) = F_2(g^2, p^2)$ $F_2(g^1, p^1) = F_1(g^1, p^1) + p^1_j = f_0(A_1) +$ p^1_j **. On the other hand, for (**g, p**)** D**, we have** $F_2(g, p) = F_1(g, p) + p_j$ $F_1(g^1, p^1) + p_j$**, whence, with (**g, p**) = (**g^2, p^2**), we obtain** $f_0(A_2)$ $f_0(A_1) + p^2_j$ **. Therefore,** p^2_j $d + y_{fj}$ p^1_j **. The price systems** p^k $P(A_k)$ **were chosen arbitrarily, so the theorem is proved.** *Q.E.D.*

Lemma 18. **Let** $e = (z, p)$ **be an equilibrium for situation** A_2 **and** l **=** j**(1),** **,** $j(n+1)$ **a sequence actualized in** e**. (a) If** $j(s+1)$ j**, then** $d_{j(s)}(p)$ $d_{j(s+1)}(p)$**. (b) If** l **is a cycle, then all** $d_i(p)$ **with** i $I(l)$ **are equal. (c) If** l **is a chain and** i I $I(l)$**, then** $d_i(p)$ **0. (d) If** l **is a chain of the first type and** i $I(l)$**, then** $d_i(p) = 0$**.**

Proof. **Without loss of generality we can consider that l is a maximal by embedding actualized sequence. Put $p = c(p)$, $j(s) = j(z^1, h(s))$. Lemma 1 (being applied to the equilibria e_1 and e) gives:**

$$y_{h(s)j(s)} \quad y_{h(s)j(s+1)} \quad p^1_{j(s)} \quad p^1_{j(s+1)} \quad \text{and} \quad y_{h(s)j(s+1)} \quad y_{h(s)j(s)} \quad p_{j(s+1)} \quad p_{j(s)}. \tag{67}$$

Then $p^1_{j(s)} \quad p^1_{j(s+1)} \quad p_{j(s)} \quad p_{j(s+1)}$, whence $d_{j(s)}(p) \quad d_{j(s+1)}(p)$. The statement (a) is proved.

If l is a cycle, then (a) gives $d_{j(1)}(p) \quad d_{j(2)}(p) \qquad d_{j(n)}(p) \quad d_{j(1)}(p)$, whence (b) follows.

Let l be a chain. If $j(1) = 0$, then $d_{j(1)}(p) = 0$. Assume that $j(1) \quad 0$. Then $j(1)$ $\{j(z, g) \mid g \quad G\}$, $p_{j(1)} = 0$ by (10), and $d_{j(1)}(p) \quad 0$. Now (c) follows from (a).

Suppose l is a chain of the first type and $k = j(n+1)$. The maximality of l implies that $k \quad F$. Then $p^1_k = 0$ by (10), and Assumption 1 implies that $k \quad \{d(g) \mid g \quad G_1\}$. Assume that $g(k) = g \quad G_2$. By the definition of an actualized in e sequence, $k = j(z, h)$ for some $h \quad G$. If $h \quad G_1$, then, by Lemma 2, $p_k \quad y_{gk} \quad 0 = p^1_k$. If $h \quad G_2$, then $h = g$, $l = \ 0, k$, $j(z^1, g) = 0$, $j(z, h) = k$. Applying Lemma 2 to the equilibria e and e_1, we obtain $0 = p^1_k \quad y_{gk} \quad p_k$; consequently, $p^1_k = p_k = 0$. If $k = 0$, then $p^1_k = p_k = 0$ once more. So, $d_k(p) \quad 0$ in any case. Now (a) and (c) imply $0 \quad d_{j(1)}(p) \qquad d_{j(n+1)}(p) \quad 0$, whence (d) follows. *Q.E.D.*

Lemma 19. If (z, p) is some equilibrium for situation A_2 and $j \quad I(l_g(z))$, then $v_{gd(g)}(p) = u_g(z)$.

Proof. **Suppose $a = j(z^1, g) = d(g)$, $b = j(z, g)$, $j \quad I(l_g)$. It follows from (67) that $(y_{ga} \quad y_{gb}) \quad (p^1_a \quad p^1_b) \quad 0 \quad (y_{ga} \quad y_{gb}) \quad (p_a \quad p_b)$. But l_g is either a chain of the first type or a cycle, thus $d_a = d_b$ (Lemma 18), whence $p^1_a \quad p^1_b = p_a \quad p_b$. Then $0 = (y_{ga} \quad y_{gb}) \quad (p_a \quad p_b) = v_{ga}(p) - u_g(z)$.** *Q.E.D.*

Theorem 12. If $(z, p) \quad$ SE(l) and $\bar{z}_g = (y_{gj(z,g)}(\bar{p}), j(z, g))$, then $(\bar{z}, \bar{p})$ SE(l).

Proof. Put $\bar{p}$ = $c(\bar{p})$. Let us show that $(\bar{z}, \bar{p})$ SE(I). The equilibria (z, p) and $(\bar{z}, \bar{p})$ create the same allocation, thus $(\bar{z}, \bar{p})$ satisfies condition (10). By Lemma 1, it is enough to check for $(\bar{z}, \bar{p})$ the condition:

$$y_{gj(z,g)} \quad \bar{p}_{j(z,g)} \quad y_{gi} \quad \bar{p}_i \quad \text{for } g \quad G \quad \{g(j)\},\ i \quad K_g. \tag{68}$$

This condition is true for $g(j)$ since $\bar{p}_i = p_i$ for i $K_{g(j)}$. Let us take some g G and put $a = j(z^1, g)$, $b = j(z, g)$. Since e^1 and e are equilibria for situations A_1 and A_2 respectively, the following inequalities are true (Lemma 1):

$$y_{ga} \quad p_a^1 \quad y_{gi} \quad p_i^1 \quad \text{for } i \quad J_g \quad \text{and} \quad y_{gb} \quad p_b \quad y_{gi} \quad p_i \quad \text{for } i \quad K_g. \tag{69}$$

Consider g G_1, then $K_g = I$ $\{j\}$. Assume that b $I(I)$; then a $I(I)$. By Lemma 18, $\bar{p}_b = p_b$ and $\bar{p}_a = p_a$ p_a^1. Using (69), we obtain y_{gb} p_b y_{ga} p_a y_{ga} p_a^1 y_{gi} p_i^1 for i I. Whence, taking into account the second inequality in (69), we have (68) for i I (since $\bar{p}_i$ $\{p_i^1, p_i\}$); (68) for $i = j$ follows from (69) and $\bar{p}_j = p_j$. Assume now that b $I(I)$. Then $b = a$, y_{gb} $\bar{p}_b = \max\{y_{gb}$ p_b, y_{gb} $p_b^1\}$ by the definition of $\bar{p}$. If i I, then (69) implies y_{gb} $\bar{p}_b$ $\max\{y_{gi}$ p_i^1, y_{gi} $p_i\} = y_{gi}$ $\bar{p}_i$. If $i = j$, then (69) gives y_{gb} $\bar{p}_b$ y_{gj} $p_j = y_{gj}$ $\bar{p}_j$.

Consider g G_2, then $J_g = K_g = \{0, d(g)\}$ and $y_{gd(g)}$ 0 by the definition. If $j(z, g) = d(g)$, then $p_{d(g)}$ $y_{gd(g)}$ by Lemma 2, hence $\bar{p}_{d(g)}$ $y_{gd(g)}$ as well. Assume that $j(z, g) = 0$, then $p_{d(g)}$ $y_{gd(g)}$ by Lemma 2. If $d(g)$ $\{j(z, h) \mid h$ $G_1\}$ $\{j(z^1, h) \mid h$ $G_1\}$, then $p_{d(g)}^1$ $y_{gd(g)}$ by Lemma 2, thus $\bar{p}_{d(g)}$ $y_{gd(g)}$. If $d(g)$ $\{j(z, h) \mid h$ $G_1\}$, then $p_{d(g)} = 0$, whence $\bar{p}_{d(g)} = y_{gd(g)} = 0$. In any case condition (68) is fulfilled. *Q.E.D.*

Theorem 13. $e(I, \hat{p})$ SE(I), and $\hat{p}_i$ p_i^1 for i I.

Proof. We claim that $e(I, \hat{p})$ is an equilibrium. It is enough to check the conditions $\hat{p}$ 0, $\hat{p}_0 = 0$, (9), and (10). If $g(i)$ G_2, then there is the path m = 0, i in graph G, and $\hat{p}_i$ $D(m) = y_{g(i)i}$ 0. If $g(i)$ G_1, then $j(z^1, g) = i$, there is the path m = i, i in graph G, and $\hat{p}_i$ $D(m) = 0$. Finally, there is the path m =

0, 0 in graph G, thus $\hat{p}_0$ $D(m) = 0$. On the other hand, m_0 is either a chain of the first type or cycle, therefore $\hat{p}_0 = D(m_0)$ 0 by Theorem 2 (since (z^1, p) is an equilibrium). Hence, $\hat{p}$ 0 and $\hat{p}_0 = 0$.

Suppose i 0 and i $\{j(l, g) \mid g \; G\}$. Then i F $\{j(1)\}$. If $D(m_{j(1)}) > 0$, then the weight of the path obtained by attachment of $m_{j(1)}$ to l from the left is $D(m_{j(1)}) + D(l) > D(l)$, contrary to the choice of l. Thus $\hat{p}_{j(1)} = \hat{p}_{j(1)} = 0$. If i F, then m_i is a chain of the first type, and $D(m_i)$ 0, since (z^1, p) is an equilibrium. Whence, taking into account $\hat{p}$ 0, it follows that $\hat{p}_i = \hat{p}_i = 0$. So, (10) is true for $e(l, \hat{p})$.

Consider some g G and put $a = j(z^1, g)$, $b = z_g(l) = j(l, g)$. By Lemma 1, condition (9) for $e(l, \hat{p})$ is equivalent to $y_{gb} - \hat{p}_b$ y_{gi} $\hat{p}_i$ for all i K_g. Assume the contrary: i K_g and

$$\hat{p}_i < \hat{p}_b \quad y_{gb} + y_{gi}. \tag{70}$$

Suppose b $I(l)$. Then a $I(l)$ and, by the choice of l, $\hat{p}_b = \hat{p}_a + y_{gb}$ y_{ga}. From this and (70) we obtain $\hat{p}_i < \hat{p}_a + y_{gi}$ $y_{ga} = \hat{p}_a + c(a, i)$ or $D(m_i) < D(m_a) + c(a, i)$, contrary to the definition of m_i. Suppose now b $I(l)$. Then $a = b$ and (70) gives $\hat{p}_i < \hat{p}_a + c(a, i)$, this leads to contradiction, as in the previous case. So, $e(l, \hat{p})$ is an equilibrium, and $e(l, \hat{p})$ SE(l) by construction.

If $\hat{p}_i = 0$, then $\hat{p}_i$ p_i^1; in particular, $\hat{p}_{j(1)} = 0$ (this was proved above) implies $\hat{p}_{j(1)}$ $p_{j(1)}^1$. Suppose i $I \setminus \{j(1)\}$, $\hat{p}_i$ 0, and $m_i =$ $i(1), i(2),$ $, i(n) = i$. If $\hat{p}_{i(1)} >$ 0, then some path in graph G is greater than $D(m_i)$ in weight, contrary to the choice of $D(m_i)$. Thus $\hat{p}_{i(1)} = 0$. The weight of m_i cannot be negative, and an initial segment of zero weight can be excluded from m_i; at the same time, $\hat{p}_{i(s)} = D($ $i(1),$ $, i(s)$ $)$. Therefore, we can consider $\hat{p}_{i(s)} > 0$ for $s > 1$ (in particular, $i(2)$ 0). Then $i(s)$ $\{z_g(l) \mid g \; G\}$ $\{j(z^1, g) \mid g \; G\}$ for $s > 1$ (in other words, dwelling $i(s)$ is chosen by some agent in equilibrium e^1). Suppose that $i(s) = j(z^1, g_s)$, $s > 1$. Lemma 1 for equilibrium e^1 implies $c(i(s), i(s+1))$

$p^{1}_{i(s+1)} \quad p^{1}_{i(s)}$. **Summing these inequalities over** $s > 1$**, we get:**

$$\hat{p}_i \quad \hat{p}_{i(2)} = D(m_i) \quad c(i(1), i(2)) \quad p^1_i \quad p^1_{i(2)} . \qquad (71)$$

If $i(1) \quad \{j(z^1, g) \mid g \quad G\}$**, then, as above, Lemma 1 implies** $\hat{p}_i \quad \hat{p}_{i(1)} \quad p^1_i \quad p^1_{i(1)}$ p^1_i. **But** $\hat{p}_{i(1)} = 0$**, thus** $\hat{p}_i \quad p^1_i$. **If** $i(1) \quad \{j(z^1, g) \mid g \quad G\}$**, then the arc** $(i(1), i(2))$ **has the form** $(0, d(g))$ **for some** $g \quad G_2$ **(since** $i(2) \quad 0$**). Then** $\hat{p}_{i(2)} = c(0, d(g)) =$ $y_{gd(g)} \quad p^1_{i(2)}$ **by Assumption 10, and (71) implies** $\hat{p}_i \quad p^1_i$. *Q.E.D.*

Corollary 1. **If** $i \quad I$**, then** $\hat{p}_i \quad \min\{p_i \mid p \quad P(A_1)\}$**.**

Proof. **Suppose** $p \quad P(A_1)$**. Then there exists some distribution** z **for situation** A_1 **such that** $z(z) = z(z^1)$ **and** (z, p) **is an equilibrium for this situation (because each equilibrium price system equilibrates any equilibrium allocation, see comments to Theorem 2). The graph** G **is completely determined by allocation** $z(z^1)$ **and** G **does not depend on prices, thus we can replace** p^1 **by** p **in Theorem 13.** *Q.E.D.*

Corollary 2. **If** $(z, p) \quad \mathrm{SE}(I)$**, then** $p_i \quad \hat{p}_i$ **for** $i \quad I(I)$**.**

Proof. **Suppose** $e = (z, p) \quad \mathrm{SE}(I)$**; put** $I(k) = \quad j(1), \quad , j(k)$ **(an initial segment of** I**) and** $p = c(p)$**. The path** I **has the maximal weight, thus** $\hat{p}_{j(k)} = D(I(k))$ **for** $k \quad n+1$**. In equilibrium** e^1**, dwelling** $j(s)$**,** $s \quad n$**, is chosen by some agent** $h(s)$**:** $j(s) = j(z^1, h(s))$**. In equilibrium** e**, this agent chooses** $j(s+1)$**. Lemma 1 for** e **implies** $c(j(s), j(s+1)) \quad p_{j(s+1)} \quad p_{j(s)}$**. Summing these inequalities over** s **from 1 to k 1, we get** $p_{j(k)} \quad p_{j(1)} \quad D(I(k)) = \hat{p}_{j(k)}$ **for** $k \quad n+1$**. By (10), it follows from** $j(1) \quad \{z_g(I) \mid g \quad H\}$ **that the dwelling** $j(1)$ **has zero price in all equilibria from** $\mathrm{SE}(I)$**. Then** $p_{j(k)} \quad \hat{p}_{j(k)}$ **for** $k > 1$**. And** $p_{j(1)} = \hat{p}_{j(1)} = 0$ **since** $e(I, \hat{p}) \quad \mathrm{SE}(I)$ **(Theorem 13).** *Q.E.D.*

Corollary 3. **If** $i \quad I$**,** $k \quad I(m_i) \setminus \{i\}$**, and** $\hat{p}_k < p^1_k$**, then** $\hat{p}_i < p^1_i$**.**

Proof. **Suppose $m_i =\ i(1), i(2),\ \ , i$ and put $m_i(s) =\ i(1), i(2),\ \ , i(s)$. The definition of the path m_i implies that $\hat{p}_{i(s)} = D(m_i(s))$. The corresponding components of the vectors $\hat{p}\ \ p^1$ and $\hat{p}\ \ p^1$ have the same sign. Thus $\hat{p}_k < p^1_k$ and it is enough to show that $\hat{p}_i < p^1_i$. If $k = i(1)$ and $k\ \ \{j(z^1, g) \mid g\ \ G\}$, then $p^1_k = 0$ by (10) and $\hat{p}_k\ \ p^1_k$ contrary to hypothesis of the corollary. Thus $k = i(s)$ and either $s > 1$ or $k\ \ \{j(z^1, g) \mid g\ \ G\}$. Let us put $m =\ i(s), i(s+1),\ \ , i$. The arguments analogous to those used in the proof of inequality (71) give us $D(m)\ \ p^1_i\ \ p^1_{i(s)}$. Then $\hat{p}_i\ \ \hat{p}_{i(s)} = D(m)\ \ \ p^1_i\ \ \ p^1_{i(s)}$, whence $p^1_i\ \ \ \hat{p}_i\ \ \ p^1_{i(s)}$ $\hat{p}_{i(s)} > 0$ under hypothesis. .** *Q.E.D.*

Theorem 14. In situation A, $e(\tilde{p})$ is an equilibrium. If $p\ \ P(A)$ and $c(p_{d(g)})$ $y_{gd(g)}$ for all $g\ \ G_2$, then $\tilde{p}\ \ \ p$.

Proof. **It is enough to prove conditions $\tilde{p}\ \ \ 0$, $\tilde{p}_0 = 0$, (9), and (10). If $g(i)$ G_2, then there is the path $m =\ 0, i$ in graph $G(z)$ and $\tilde{p}_i\ \ \ D(m) = y_{g(i)i}\ \ \ 0$. If $g(i)\ \ G_1$, then there is the path $m =\ i, i$ in graph $G(z)$ and $\tilde{p}_i\ \ D(m) = 0$. Finally, there is the path $m =\ 0, 0$ in graph $G(z)$, therefore $\tilde{p}_0\ \ \ D(m) = 0$. On the other hand, the path m_0 with the last vertex zero cannot be of positive weight by Theorem 2 (since z is an equilibrium allocation). Consequently, $\tilde{p}\ \ \ 0$ and $\tilde{p}_0 = 0$.**

Assume that $i\ \ \ 0$ and $i\ \ \{z(g) \mid g\ \ G\}$. Then, as in the case of $i = 0$, $D(m_i)$ 0 by Theorem 2. Whence, taking into account $\tilde{p}\ \ \ 0$, we obtain $\tilde{p}_i = 0$. So, condition (10) is true for $e(\tilde{p})$.

Take some $g\ \ G$ and put $b = z(g)$. We have $D(m_i)\ \ \ D(m_b) + c(b, i)$ for any $i\ \ J_g$. Then $\tilde{p}_i < \tilde{p}_b\ \ \ y_{gb} + y_{gi}$, and this, by Lemma 1, is equivalent to condition (9) for $e(\tilde{p})$. Hence, $e(\tilde{p})$ is an equilibrium.

Suppose $p\ \ P(A)$ and $p = c(p)$. Take some $i\ \ I$. If $\tilde{p}_i = 0$, then $\tilde{p}_i\ \ \ p_i$. Assume

that $\tilde{p}_i > 0$ and m_i = $i(1), i(2),\ \ , i(n) = i$. A path with the last vertex zero cannot be of positive weight by Theorem 2 (since z is an equilibrium allocation), thus we can suppose that $i(k)\ \ 0$ for $k\ \ 2$. Then $i(k)\ \ \{z(g) \mid g\ \ G\}$ for $k\ \ 2$ by the definition of the graph $G(z)$. Let us assume first that $i(1)\ \ \{z(g) \mid g\ \ G\}$; then $i(k) = z(h(k))$ for $k\ \ 1$. By Lemma 1, $y_{h(k)i(k+1)}\ \ y_{hk)i(k)}\ \ p_{i(k+1)}\ \ p_{i(k)}$. Summing this inequalities over k from 1 to $n\ \ 1$, we get $\tilde{p}_i\ \ \tilde{p}_{i(1)}\ \ p_i\ \ p_{i(1)}\ \ p_i$. But $\tilde{p}_{i(1)} = 0$ (since m_i is the path of maximal weight with the last vertex i), thus $\tilde{p}_i\ \ p_i$. Assume now that $i(1)\ \ \{z(g) \mid g\ \ G\}$. Then $(i(1), i(2)) = (0, d(g))$ for some $g\ \ G_2$ and $z(g) = d(g)$. Whence it follows that the arc $(i(2), 0)$ with the weight ($y_{gd(g)}$) 0 is a unique arc with the first vertex $i(2)$ in graph $G(z)$. Thus m_i = $i(1), i(2)$, $i = i(2)$, and $\tilde{p}_i = D(m_i) = y_{gd(g)}\ \ p_i$ under hypothesis. *Q.E.D.*

Lemma 20. Suppose $e = (z, p)$ is some equilibrium for situation A_2, $p\ \ p^1$, and $g\ \ G_1$; then $u_g(z^1)\ \ u_g(z)$.

Proof. Put $p = c(p)$, $g\ \ G_1$, and $e = u_g(z)\ \ u_g(z^1)$. By condition (10), $e\ \ v_{gd(g)}(p)\ \ u_g(z^1) = [p_{d(g)}\ \ p^1_{d(g)}]\ \ [p_{d(g)}\ \ p^1_{d(g)}]$. If $p^1_{d(g)} = 0$, then $p_{d(g)} = 0$ under hypothesis, and thus $e\ \ p^1_{d(g)}\ \ p_{d(g)}\ \ 0$. If $p^1_{d(g)} > 0$, then $d(g)\ \ 0$, and Assumption 1 implies that $d(g) = d(g)$; thus $e\ \ v_{gd(g)}(p)\ \ u_g(z^1) = 0$. *Q.E.D.*

Theorem 15. Let y be an optimal solution of PDR. Then $x(y)$ is an optimal solution to problem (50) (53). Under Assumption 11, $x(y)$ determines some equilibrium allocation for situation $A(y)$.

Proof. Obviously, $x(y)$ satisfies conditions (51) (53). Let $x = (x_t(k))$ be an optimal solution to problem (50) (53). The first statement of the theorem follows from $\ _{t,k} a_t(k) x_t(k)\ \ \ _{t,k}(b_t + d_t) x_t(k) = \ _t (b_t + d_t) \times \ _k x_t(k) = \ _t (b_t + d_t) y_t = \ _{t,k} a_t(k) x_t(k, y)$. Now the second statement follows from Theorem 10 and the discussion of the ATC problem in section 4. *Q.E.D.*